BROCK UNIVERSITY

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Course: MATH 4F85	Number of students: 27
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Time of Examination: 19:00-22:00	Instructor: J. Vrbik

Full credit given for 7 correct and complete answers.

1. Customers arrive at a store at an average rate of 7.2 'clusters' per hour. The cluster sizes are independent of each other, and have a distribution with the following probability function

   $$f(i) = \binom{3}{i-1} \times 0.4^{i-1} \times 0.6^{4-i} \qquad \text{where } \ i \ \text{ is a non-negative integer smaller than 5}$$

   Find

   (a) the expected size of a 'cluster', and the corresponding standard deviation,

   (b) the probability that more than 10 customers will arrive at the store during the next eighteen minutes,

   (c) the probability that the next two clusters (combined) will yield a total of at least 4 customers.

2. For a continuous-time Markov chain with the following (per hour) rates

   $$\begin{bmatrix} \times & 2.1 & 3.5 & 1.0 \\ 0.9 & \times & 2.3 & 1.8 \\ 3.3 & 1.1 & \times & 2.0 \\ 4.1 & 0 & 2.7 & \times \end{bmatrix}$$

   (a) find the corresponding (exact, i.e. using fractions) stationary distribution.

   (b) If the process is currently in the last state, what is the probability that, 13 minutes later, the process will still be in the last state (has not left it yet)?

   (c) If the process is currently in the last state, what is the probability that, 13 minutes later, the process will be in the last state (this time, it may have left it and come back, possibly more than once)?

   Modify the above matrix to make the first state absorbing.

   (d) If the process is currently in the last state, what is expected time till absorption, and the corresponding standard deviation?

3. Consider a Birth and Death process with the following (per minute) rates

   $$\begin{aligned} \lambda_n &= 3.7 + n \\ \mu_n &= 1.3n \end{aligned}$$

   Is this process stationary? If yes, find the expected value and standard deviation of the stationary distribution. In the long run, how often is State 10 visited, on the average, per hour? If the process has the value of 5 now, what is the probability it will have a value bigger than 7 ten minutes later?

4. Consider a Birth and Death process with the following (per minute) rates

$$\begin{aligned}\lambda_n &= 1.04 \times (1.03^n - 1) \\ \mu_n &= 1.03 \times (1.04^n - 1)\end{aligned}$$

Given that the process is now in State 10, find the probability that it will be (sooner or later) absorbed by State 0. If this probability is equal to 1, find the expected length of the time till absorption (starting in State 10).

5. Consider a Brownian motion with no drift, and the diffusion coefficient of $8.3\ \frac{\text{mm}^2}{\text{hr}}$. If the process is observed to have the value of -5 mm at 8:00 compute the probability that

   (a) for the next 2.5 hours (till 10:30), it will have only negative values,

   (b) it will have a negative value at 10:30, without ever (during those 2.5 hours) reaching 2 mm,

   (c) it will have the value of -5 mm, at least once, between 10:30 and 12:00.

   (d) Now assume that, in addition to knowing its value at 8:00 (to be -5 mm), we also know that at 12:00 it reached the value of 2 mm. Find the probability that the process had a value bigger than 5 mm at 10:30.

6. Find the eigenvalues and the corresponding constituent matrices of

$$\mathbb{M} = \begin{bmatrix} 2 & 1 & -5 & -2 \\ 2 & 7 & 1 & 2 \\ 6 & -1 & 13 & 2 \\ -2 & 0 & -2 & 6 \end{bmatrix}$$

Using these, evaluate $\ln(\mathbb{M})$.

7. Consider an M/M/2 queue with 40 customers arriving on the average every hour, but walking away with the probability of 20% when both servers are busy and no one is waiting, and with the probability of $1 - 0.6^m$ when $m \geq 1$ people are waiting (no walking away when the arriving customer can get immediate service). The average service time is 10 minutes. Compute the long-run

   (a) average number of customers waiting for service,

   (b) server utilization factor,

   (c) average number of customers served per hour,

   (d) and the average waiting time (of the customers who decided to stay). Hint: Use Little's formula.

8. Assuming that $-1 < z < 1$, solve

$$z \overset{\bullet}{P}(z,t) + (1-z^2)\ P'(z,t) = P(z,t)$$

subject to the following initial condition:

$$P(z,0) = \frac{1}{1-z}$$

9. Suppose we roll four dice.

   (a) Find the probability generating function of the total number of dots we get.

   (b) What is the probability of getting more than 9 dots (in total)?

   After we roll the four dice, we flip a coin as many times as the total number of dots obtained.

   (c) What is the expected number of heads we thus get, and the corresponding standard deviation?

   (d) What is the probability of getting more than 9 heads?

10. Assume that in a large portion of our Galaxy, stars are distributed randomly with the average density of 6.72 per 1000 $\mathtt{ly}^3$ ($\mathtt{ly}$ stands for light year).

    (a) Find the probability that a randomly selected star will have at least 15 'neighbors' whose distance (from this star) is less than 8 light years.

    (b) Find the distribution function of a distance from a randomly selected star to its 3^{rd} closest neighbor.

    (c) Find the expected value and standard deviation of the distribution in Part b.

11. Consider an $M/G/\infty$ queue with customers arriving at an average rate of 4.3 per hour, and the service time having a distribution with the following probability density function

$$f(x) := \begin{cases} \frac{3}{4}\left[1-(x-2)^2\right] & \text{when} \quad 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

(x is time in hours). At 9:30, all servers are idle.

   (a) What is the probability that, by 12:00, more than 2 customers have completed their service (since 9:30) while exactly 8 are being serviced?

   (b) Find the mean and standard deviation of the stationary distribution of the number of customers in the system.

   (c) How often, in the long run, does it happen that all servers become idle (give the average number of such occurrences per 365 days).