Final Examination: April 2010 | Number of Pages: 4
Course: MATH 4F85 | Number of students: 4
Date of Examination: Apr. 20, 2010 | Number of Hours: 3
Time of Examination: 19:00-22:00 | Instructor: J. Vrbik

Two sheet of notes, and use of Maple, are allowed.

No examination aids other than those specified on the examination scripts are permitted (this regulation does not preclude special arrangements being made for students with disabilities). Translation dictionaries (e.g. English-French) or other dictionaries (thesaurus, definitions, technical) are not allowed unless specified by the instructor and indicated on the examination paper.

Full credit given for 7 complete answers.

Numerical answers must be correct to 4 significant digits.

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1. Consider the basic Poisson process of 'arrivals' with $\lambda = 13.7$ per hour, which is observed for a random time T, where T has the $\mathsf{gamma}(4, 36 \text{ minute})$ distribution. Compute
   (a) $\Pr(T > 2 \text{ hr})$,
   (b) the probability that, during time T, the process will experience more than 50 arrivals,
   (c) the expected number of arrivals and the corresponding standard deviation.

2. For a continuous-time Markov chain with the following (per hour) rates

$$\begin{bmatrix} \times & 2.1 & 3.5 & 1.0 & 0.8 \\ 0.9 & \times & 2.3 & 1.8 & 2.0 \\ 3.3 & 1.1 & \times & 2.0 & 1.7 \\ 4.1 & 0 & 2.7 & \times & 0.4 \\ 0.7 & 1.2 & 1.3 & 2.1 & \times \end{bmatrix}$$

   (a) find the corresponding (exact, i.e. using fractions) stationary distribution.
   (b) If the process is currently in the last state, what is the probability that, 17 minutes later, the process will be in the first state?
   (c) If the process is currently in the last state, what is the probability that, 17 minutes later, the process will be in the first state, without ever (during these 17 minutes) visiting the third (middle) state?

3. Consider a LGWI process describing a colony of 'bacteria' with the initial value 75. The expected lifetime of each bacterium is 2 minutes and 35 seconds, each living bacterium produces an 'offspring' at the average rate of 1 every 3 minutes, and the rate of immigration is 2.3 per minute.

   (a) Spell out the corresponding expressions for λ_n and μ_n (make sure you get these correct, before you continue).

   (b) What is the probability of having fewer than 50 bacteria 9 minutes later?

   (c) Find the expected value and standard deviation of the time of death of the last of the initial members.

   (d) Find the expected value of the time till extinction of the 'native' subpopulation (the initial 75 bacteria, and their progeny).

4. Consider a Birth and Death process with the following (per minute) rates

$$\begin{aligned} \lambda_n &= 0.6\sqrt{n} \\ \mu_n &= \frac{3n}{1+n} \end{aligned}$$

   Given that the process is now in State 30, find the probability that

   (a) it will become extinct (reaching State 0),

   (b) the next three transitions will be all 'births',

   (c) no transition will take place during the next 22 seconds.

5. Consider a Brownian motion with no drift, and the diffusion coefficient of 12 $\frac{\text{mm}^2}{\text{hr}}$. If the process is observed to have the value of 3.2 mm at 8:23, compute the probability that

   (a) it will have a value smaller than -2 mm at 10:47,

   (b) it will have reached the value of 5.3 mm (at least once) before 10:47,

   (c) it will have a value smaller than -2 mm at 10:47, without ever (during that interval) reaching 5.3 mm.

   (d) it will have returned to the value of 3.2 mm (at least once) after 10:47 but before 12:00.

6. Find the eigenvalues and the corresponding constituent matrices of

$$\mathbb{M} = \begin{bmatrix} 11 & 19 & 15 & 4 \\ 7 & 5 & 9 & -1 \\ -12 & -15 & -15 & -6 \\ -2 & -7 & -3 & -7 \end{bmatrix}$$

   Using these, evaluate $\tan(\mathbb{M})$.

7. Consider an $M/M/7$ queue with the arrival rate of 15.2 customers per hour, and the mean service time of 13 minutes. Compute the long-run (average):

   (a) server utilization factor,

   (b) number of customers served per hour,

   (c) frequency of visits to State 4 (with four servers busy),

   (d) waiting time (in seconds).

8. Assuming that $0 < z < 1$, solve

$$z^2 \overset{\bullet}{P}(z,t) = z\ P'(z,t) + P(z,t)$$

   subject to the following initial condition:

$$P(z,0) = z^3$$

9. Consider a B&D process with the following (per minute) rates:

$$\begin{aligned} \lambda_n &= 98 - 7n \\ \mu_n &= 5.6n \end{aligned}$$

   and the initial value of 12. Find

   (a) the probability that, 4 seconds later, the process has a value smaller than 10,

   (b) the probability that, 4 seconds later, the process has not visited State 10 yet (hint: make State 10 absorbing),

   (c) the long-run frequency of visits to State 10.

10. Consider a continuous-time Markov chain with the following (per hour) rates

$$\begin{bmatrix} \times & 0 & 0 & 0 & 0 & 0 \\ 0.9 & \times & 2.3 & 1.8 & 2.0 & 0 \\ 0 & 1.1 & \times & 2.0 & 1.7 & 1.4 \\ 4.1 & 0 & 2.7 & \times & 0.4 & 2.8 \\ 2.1 & 3.2 & 0 & 0.9 & \times & 3.1 \\ 0 & 0 & 0 & 0 & 0 & \times \end{bmatrix}$$

   (States 1 and 6 are clearly absorbing). Given that the process starts in State 3, compute

   (a) the probability that it will 'get stuck' in one of the absorbing states within the next 23 minutes,

   (b) the expected time till absorption, and the corresponding standard deviation,

   (c) the exact (i.e. fractional) probability of being absorbed, sooner or later, by State 6.

11. Consider a Brownian motion with drift equal to $-2.3\,\frac{\text{mm}}{\text{hr}}$, and the diffusion coefficient of $12\,\frac{\text{mm}^2}{\text{hr}}$. If the process is observed to have the value of 3.2 mm at 8:23, the value of -1 mm at 10:41, and the value of -0.8 mm at 11:17, compute the probability that

    (a) it has had a positive value at 9:30,

    (b) it has had a positive value at 11:00,

    (c) it will have a positive value at 12:00.