BROCK UNIVERSITY

Final Examination: April 2011	Number of Pages: 3
Course: MATH 4F85	Number of students: 6
Date of Examination: Apr. 15, 2011	Number of Hours: 3
Time of Examination: 9:00-12:00	Instructor: J. Vrbik

Two sheet of notes, and use of Maple, are allowed. Full credit given for 7 complete answers. Numerical answers must be correct to 4 significant digits.

No examination aids other than those specified on the examination scripts are permitted (this regulation does not preclude special arrangements being made for students with disabilities). Translation dictionaries (e.g. English-French) or other dictionaries (thesaurus, definitions, technical) are not allowed unless specified by the instructor and indicated on the examination paper.

1. Consider a Poisson process with the average arrival rate of 12.3 per hour, observed for a *random* time of duration T, whose distribution is $\mathsf{gamma}(3,\ 37\text{ min}.)$. Find:

   (a) the probability generating function of $X(T) \equiv$ total number of arrivals during time T,

   (b) the expected value of $X(T)$ and the corresponding standard deviation,

   (c) $\Pr[X(T) > 23]$.

2. A time-continuous Markov chain with 6 states (labelled 1 to 6; the first and last are *absorbing*), has the following transition rates (per hour):

From ↓ To →	1	2	3	4	5	6
2	0.9	×	2.3	1.8	1.0	1.4
3	3.3	1.1	×	2.0	0	2.5
4	4.1	0	2.7	×	0.9	3.0
5	2.0	1.4	0.7	3.0	×	2.3

   If the process starts in State 3, find

   (a) the probability that, 19 minutes later, the process is still in State 3 (without ever leaving it),

   (b) the probability that, 19 minutes later, the process is in State 3 (regardless of how many transitions it has made in between),

   (c) the probability of being absorbed (sooner or later) by State 6,

   (d) expected time till absorption (in either absorbing state) and the corresponding standard deviation.

3. Consider an M/M/2 queue with 15.3 arrivals per hour (on the average), and the mean service time of 4 min 46 sec. The probability that an arrival *who has to wait for service* joins the system is 0.79^{k+1}, where k is the number of customers waiting (please realize that $k = 0$ implies two distinct possibilities, depending on how many servers are busy). Find the long-run

(a) server utilization factor,

(b) percentage of lost customers,

(c) average waiting time,

(d) frequency of visits to State 0.

4. Consider a Brownian motion without drift, diffusion coefficient equal to 5.7 $\frac{\text{mm}^2}{\text{hr}}$, and an absorbing barrier at -1.3 mm. Assuming that $X(8{:}24) = 6$ mm, compute the probability of

   (a) 3 mm $< X(11{:}02)$,

   (b) $X(11{:}02) <$ 3 mm, but not yet absorbed,

   (c) $\mathbb{E}[X(11{:}02)]$.

5. Compute $\ln(\mathbb{A})$, where

$$\mathbb{A} = \begin{bmatrix} 0 & -1 & -2 & 0 & 2 \\ \frac{13}{3} & 8 & \frac{10}{3} & \frac{8}{3} & -10 \\ 2 & 2 & 5 & 0 & -4 \\ 1 & 1 & 2 & 1 & -2 \\ \frac{8}{3} & 4 & \frac{8}{3} & \frac{4}{3} & -5 \end{bmatrix}$$

   Using Maple's exponential, verify your answer.

6. Consider a Birth and Death process with the following rates (per minute)

State:	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
λ_n	3.3	4.1	0.9	2.5	0
μ_n	0	3.0	4.1	1.2	2.3

   (a) Find the corresponding stationary distribution (in exact fractions).

   (b) If the process is in State $\underline{3}$ now, what is the probability of being in State $\underline{4}$, 12 minutes later? Hint: Treat it as a special case of TCMC.

   (c) In a long run, how often is State $\underline{4}$ visited, on the average, each day?

7. Solve

$$ze^z \dot{P}(z,t) = zP'(z,t) + P(z,t)$$

   subject to the following initial condition:

$$P(z,0) = z$$

8. Consider a Brownian motion with no drift and the diffusion coefficient equal to 5.3 $\frac{\text{mm}^2}{\text{hr}}$. The process has been observed to have the value of -2.5 mm at 7:30 and the value of 0.4 mm at 9:45. Find the probability of the process

   (a) having a value between -0.3 mm and 1.6 mm at 12:00,

   (b) having a value between -0.3 mm and 1.6 mm at 9:00,

   (c) avoiding the value of 0.4 mm during the period lasting from 10:45 till 11:30.

9. Consider a process consisting of 'bacteria' which, *individually*, procreate (i.e. split in two) at the rate of 0.37 per day, and have an average life span of 1 days and 7 hours (exponentially distributed). We start with a colony of 7 individuals, who are replenished by random 'immigration' at an average rate of 2.1 per day.

   (a) What is the probability that, four days later, the process consists of more than 12 bacteria?

   (b) What is the long-run average number of bacteria, and proportion of time with no bacteria at all?

   (c) What is the long-run average number of *living* immigrants (not counting their descendents).

10. This is a continuation of the previous question.

   (a) We know that the 'native population' (of the original 7 bacteria *and their progeny*) must become (sooner or later) extinct. Find the expected time till such extinction.

   (b) What is the probability that this extinction will take more than 5 days.

   (c) Also, find the expected time till the death of the last of the *original* seven bacteria (no longer counting the descendents), and the corresponding standard deviation.

11. Consider an $M/G/\infty$ queue with customers arriving at an average rate of 6.3 per hour, and the service time having a distribution with the following probability density function

$$f(t) = \begin{cases} \frac{1}{20}\exp(1-\frac{t}{20}) & 20 < t \\ 0 & \text{otherwise} \end{cases}$$

where t is time in *minutes*. At time zero, there are no customers in the system. Compute

   (a) the probability that, 49 minutes later, fewer than 3 people are being serviced, while more than 3 have already left,

   (b) the long-run average of busy servers.

   (c) In the long run, how often (on the average, per day) does the system enter State 0 (no customers)?