BROCK UNIVERSITY

Final Examination: April 2014
Course: MATH 4P85
Date of Examination: Apr. 16, 2013
Time of Examination: 14:00-17:00

Number of Pages: 4
Number of students: 14
Number of Hours: 3
Instructor: J. Vrbik

Use of Maple is allowed.

No examination aids other than those specified on the examination scripts are permitted (this regulation does not preclude special arrangements being made for students with disabilities). Translation dictionaries (e.g. English-French) or other dictionaries (thesaurus, definitions, technical) are not allowed unless specified by the instructor and indicated on the examination paper.

Full credit given for 22 (out of 33) complete answers.

Numerical answers must be correct to at least 4 significant digits.
Proper units must be given whenever appropriate.
All answers must be entered in the exam booklet.
Maple work may be printed and attached, but won't be marked by itself.

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1. Consider a Brownian motion with no drift, and the diffusion coefficient of $13.4\ \frac{\text{mm}^2}{\text{hr}}$. If the process is observed to have the value of 2.8 mm at 8:24, compute the probability that it will

   (a) pass through the value of 5.1 mm (at least once) before 10:17,

   (b) have a value smaller than 1.1 mm at 10:17, without ever (from now till then) reaching 5.1 mm.

   (c) return to 2.8 mm (at least once) between 10:17 and 11:42.

2. Consider a B&D process with States 0 to 5 and the following (per hour) transition rates

$$\begin{aligned}\lambda_n &= (n+3)\cdot\left(1-\frac{n}{5}\right)\\ \mu_n &= \frac{n}{13}\cdot(n+5)\end{aligned}$$

   The process is currently in State 3. Find (note that decimal answers are OK):

   (a) the probability of visiting State 0 before visiting State 5,

   (b) the probability of being, 17 minutes later, in State 2, without ever (during those 17 minutes) visiting either State 0 or State 5,

   (c) the long-run average time between consecutive visits to State 3 (from departure to entry).

3. Consider a (miniature) nuclear reactor with a 'population' of (originally) 306 neutrons which (*individually*), at an average rate of 0.41 per second, split a uranium atom, thus creating an extra neutron. At the same time, any neutron (including those newly created) may be absorbed by a cadmium atom and cease to exist (they do this, individually, at the rate of 0.49 per second). Furthermore, the neutrons are also replenish (due to cosmic radiation) at an average rate of 21 new neutrons per second.

   (a) Find the expected value and standard deviation of the number of neutrons 4 seconds later.

   (b) What is the long-run average number of neutrons, and the proportion of time with more than 300 neutrons?

   Note: In Part (c) assume that we eliminate cosmic radiation (a lead shield will do it), starting with the same 306 neutrons.

   (c) Find the expected time till extinction (no neutrons left).

4. Consider a Brownian motion with a drift of 3.4 $\frac{\text{mm}}{\text{hr}}$ and a diffusion coefficient of 6.3 $\frac{\text{mm}^2}{\text{hr}}$. Evaluate:

   (a) $\Pr\{X(9{:}18) < 35\,\text{mm} \mid X(8{:}00) = 31.2\,\text{mm} \cap X(9{:}00) = 33.0\,\text{mm}\}$,

   (b) $\Pr\{X(9{:}18) < 35\,\text{mm} \mid X(9{:}00) = 33.0\,\text{mm} \cap X(10{:}00) = 37.9\,\text{mm}\}$,

   (c) $\Pr\{X(9{:}18) < 35\,\text{mm} \mid X(10{:}00) = 37.9\,\text{mm} \cap X(11{:}00) = 40.6\,\text{mm}\}$.

5. Customers arrive at a 'looney' store (every item costs \$1) at an average rate of 24.2 per hour. Each customer spends a random (from our point of view) number of dollars, the distribution of which is Binomial with parameters $n = 7$ and $p = 0.31$ (note that this means that some customers, called 'browsers', will not buy a single item). Find

   (a) the probability that out of the customers who arrive between 8:21 and 9:06, there are no more than 3 browsers,

   (b) the expected value and standard deviation of the total amount of money spent by customers who arrive between 8:21 and 9:06,

   (c) the probability that this amount (from Part b) is bigger than \$35.

6. Assume that

$$\mathbb{A} = \begin{bmatrix} 3 & -2 & -4 & 0 & 2 \\ 3 & -3 & -5 & -1 & 3 \\ -1 & 1 & 3 & 0 & -1 \\ -2 & 4 & 6 & 2 & -3 \\ 1 & -2 & 0 & -1 & 2 \end{bmatrix}$$

   (a) Compute $\ln(\mathbb{A})$.

   (b) Using ‘MatrixExponential’, verify the previous answer.

   (c) Find $\mathbb{A}^n$, where n can have any value.

7. Consider an $M/M/7$ queue with the arrival rate of 29.2 customers per hour, and the mean service time of 12 minutes and 35 seconds. Compute the long-run average length (in hours, minutes and seconds) of the:

   (a) customer waiting time,

   (b) zero-utilization (no server working) *cycle* (i.e. from one entry to the next),

   (c) full-utilization (all servers working) cycle.

8. Consider a B&D process with $\lambda_n = 7.3 \times n$ per hour, $\mu_n = 7.3 \times n$ per hour, and $X(0) = 4$. Compute the probability that

   (a) $3 \le X(18 \text{ min.}) < 11$,

   (b) State 14 is never reached,

   (c) extinction will take more than 5 hours.

9. Consider an M/M/1 queue with 15.3 arrivals per hour (on the average), and the mean service time of 16 min 28 sec. The probability that a new arrival joins the system is $\dfrac{0.82^n}{1+n}$, where n is the number of customers ahead of him (*including* the one who is in service). Find the long-run

   (a) percentage of lost customers,

   (b) average length of the line up,

   (c) proportion of time with more than 3 customers *waiting* for service.

10. Consider a 2D Poisson process with the average density of 'points' equal to 1.19 per meter^2. Find

    (a) the expected distance to the 4^{th} nearest point from the origin, and the corresponding standard deviation,

    (b) the expected number of points inside the triangle with the following vertices: $(-2, 4)$, $(1, -3)$ and $(3, 2)$ - all in meters - and the corresponding standard deviation.

    (c) Given that there are exactly 28 points inside this triangle, what is the conditional probability that more than 5 of them are inside the square with corners at $(0, 0)$, $(0, 2)$, $(2, 2)$ and $(2, 0)$?

11. Consider a CTMC with five states (1 to 5) and the following (per minute) transition rates

$$\begin{bmatrix} \times & 2 & 4 & 3 & 1 \\ 5 & \times & 1 & 0 & 2 \\ 4 & 3 & \times & 1 & 6 \\ 2 & 2 & 3 & \times & 5 \\ 4 & 0 & 1 & 5 & \times \end{bmatrix}$$

Assuming that the process is now in State 4, find the *exact* (not decimal) value of the

(a) probability of visiting State 2 *before* the next return to State 4 (hint: use the total-probability formula based on the result of the next transition),

(b) expected time (and the corresponding *variance*) of the first entry (from now) to State 1,

(c) long-run frequency of visits to State 3.