BROCK UNIVERSITY

Final Examination: April 2016
Course: MATH 4F85
Date of Examination: Apr. 19, 2016
Time of Examination: 19:00-22:00

Number of Pages: 4
Number of students: 11
Number of Hours: 3
Instructor: J. Vrbik

Open book exam. Use of Maple is also allowed.

Full credit given for 20 (out of 32) complete answers.
Answers must be given in the decimal form (unless specified otherwise) and be correct to 4 significant digits. They must include the corresponding units (hours, inches, etc.) when appropriate.

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1. Consider a Poisson process of 'arrivals' at an average rate of 17.3 per hour, which is observed for a random time T having a distribution with the following CDF (t is in hours):

$$F_{\mathrm{T}}(t) = \begin{cases} 0 & t < 0 \\ \dfrac{1-e^{-t}}{1-e^{-1}} & 0 \le t < 1 \\ 1 & 1 \le t \end{cases}$$

   Compute

   (a) $\Pr(T > 42 \text{ minutes})$,

   (b) the probability that, during time T, the process will experience fewer than 9 arrivals,

   (c) the expected number of arrivals during time T, and the corresponding standard deviation.

2. For a continuous-time Markov chain with the following (per hour) transition rates

$$\begin{bmatrix} \times & 2 & 2 & 1 & 1 & 3 \\ 1 & \times & 2 & 0 & 2 & 1 \\ 3 & 1 & \times & 2 & 1 & 0 \\ 4 & 0 & 2 & \times & 1 & 2 \\ 1 & 0 & 1 & 2 & \times & 2 \\ 2 & 1 & 2 & 0 & 1 & \times \end{bmatrix}$$

   (a) find the corresponding (exact, i.e. using fractions) stationary distribution.

   (b) If the process is currently in State 5, what is the probability that, 14 minutes later, the process will be in State 3, without ever (during those 14 minutes) visiting State 1?

   (c) If the process is currently in State 5, what is the expected time (and the corresponding standard deviation) it will take it to reach (for the first time) State 1?

3. Consider Brownian motion with no drift and the diffusion coefficient of 13 $\frac{\text{mm}^2}{\text{hr}}$. If the process has been observed to have the value of 3.6 mm at 6:18 and 2.3 mm at 8:36, compute the probability that

   (a) it will reach the value of 4.8 mm (at least once) before 10:27,

   (b) it will end up with a value smaller than 1.9 mm at 10:27, without ever (during the 8:36 to 10:27 interval) reaching 4.8 mm.

   (c) it will return to the value of 2.3 mm (at least once) after 10:27 but before 11:53,

   (d) it had a value smaller than 1.9 mm at 7:07.

4. Consider a Birth and Death process with the following (per *minute*) rates

$$\begin{aligned} \lambda_n &= \frac{2n}{1+n} \\ \mu_n &= \frac{3\sqrt{n}}{1+\sqrt{n}} \end{aligned}$$

   Given that the process is now in State 18, find

   (a) the expected time till extinction,

   (b) the probability that the process never reaches State 28,

   (c) the probability that, 3 transitions (i.e. births or deaths) later, the process will be in State 17.

5. Consider a $M/G/\infty$ queue with the arrival rate of 6.7 customers per hour, the service-time distribution being gamma(3, 9 minutes) and no customers (either waiting or being serviced) at 8:07. Compute

   (a) the probability that, by 9:23, exactly 8 customers have arrived, of whom exactly 5 have already left,

   (b) the long-run proportion of time with at least 4 customers being serviced,

   (c) the probability that, while the first customer (who shows up after 8:07) is being serviced, no other customers will arrive (hint: remember PP of random duration).

6. Express

$$\exp\left(\begin{bmatrix} 12 & 0 & -16 & 18 \\ 5 & 3 & -10 & 10 \\ 11 & 1 & -19 & 22 \\ 4 & 1 & -9 & 11 \end{bmatrix} \cdot t\right)$$

   as a linear combination of the corresponding constituent matrices.

7. Consider an $M/M/5$ queue with the arrival rate of 14.2 customers per hour, and the mean service time of 18 minutes. Compute the long-run:

   (a) server utilization factor,

   (b) frequency of visits to State 7 (with 5 customers being serviced and 2 queueing up),

   (c) average waiting time (in minutes and seconds).

8. Without Maple, find

   (a) a general solution to

$$(1 + \tan z)\dot{P}(z,t) = P'(z,t) + 2P(z,t)$$

   (b) the solution which also meets the following condition:

$$P(z,0) = \frac{1}{\cos^2 z}$$

9. Consider a B&D process with the following (per minute) rates:

$$\begin{aligned} \lambda_n &= 6.3n \\ \mu_n &= 4.6n \end{aligned}$$

   and the initial value of 9. Find the probability

   (a) that, 15 seconds later, the process has a value bigger than 11,

   (b) of its ultimate extinction,

   (c) that the process never visits State 4 (hint: make it a new absorbing state).

10. Consider a 2D Poisson process with the average density of points of 0.32 per inch^2. Find

   (a) the expected distance to the fourth nearest point from the origin, and the corresponding standard deviation,

   (b) the expected number of points inside the region defined by

$$x^2 + 4x + y^2 - 2y < 4$$

   (x and y are rectangular coordinates, in inches), and the corresponding standard deviation,

   (c) given that there are exactly 12 points inside this (Part b) region, what is the conditional probability that at least 7 of them have the y coordinate bigger than 1. (hint: use Binomial distribution).

11. Consider a B&D process with (per *minute*) rates given by

n	0	1	2	3	4	5	6	7
λ_n	1.2	2.0	1.3	2.1	2.5	3.0	1.4	0
μ_n	0	2.3	4.2	2.5	1.3	1.9	2.2	3.1

and the initial value of 4. Find:

(a) the expected time till the first visit (entry) to State 0, and the corresponding standard deviation,

(b) the probability of visiting State 0 before visiting State 7,

(c) the long-run frequency of visits to State 4 (per hour).