BROCK UNIVERSITY

Final Examination: December 2016 | Number of Pages: 4
Course: MATH 4F85 | Number of students: 8
Date of Examination: Dec. 19, 2016 | Number of Hours: 3
Time of Examination: 16:00-19:00 | Instructor: J. Vrbik

Open book exam. Use of Maple and a calculator is allowed.
No examination aids other than those specified on the examination scripts are permitted (this regulation does not preclude special arrangements being made for students with disabilities).

Full credit given for 20 (out of 34) correct and complete answers.
Numerical answers must be correct to 4 significant digits.

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1. Consider an $M/G/\infty$ queue with customers arriving at an average rate of 9.3 per hour, and the service time of each customer having the gamma(7, 2 minutes) distribution. At 8:42, there are no customers in the system. Compute the

   (a) probability that, at 9:14, fewer than 5 customers are being serviced while more than 3 customers have already left,

   (b) long-run average number of busy servers

   (c) and frequency of visits to State 0 (per hour).

2. Consider a B&D process with the following (per hour) transition rates

n	0	1	2	3	4	5	6	7
λ_n	1.3	2.0	1.3	2.1	2.5	1.3	3.1	0
μ_n	0	2.3	4.2	2.2	1.3	1.9	2.8	2.0

   Find the

   (a) corresponding stationary-probability vector (in your booklet, quote its first component only),

   (b) long-run frequency of visits to State 5.

   (c) Assuming that the process is currently in State 4, what is the probability that, 21 minutes later, the process will be in State 1, without ever (during those 21 minutes) visiting State 6?

3. Consider a LGWI process with the initial value of 12. The expected lifetime of each member of this process is 17 hours, each living member splits in two on average once every 14 hours, and the rate of immigration is 5.6 members per (a 24 hour) day. Find the

   (a) expected value of the process 21 hours later, and the corresponding standard deviation,

   (b) probability of having at least 15 members 21 hours later,

   (c) probability that the native sub-population (the initial members and their descendents) will survive indefinitely,

   (d) expected number of immigrants (ignore their descendents) arriving before the time of death of the last initial member, and the corresponding standard deviation (hint: remember PP of random duration).

4. Consider a B&D process with the following (per hour) transition rates

$$\begin{aligned}\lambda_n &= 0.5\sqrt{n} \\ \mu_n &= \frac{2n}{3+n}\end{aligned}$$

   Given that the process is now in State 7, find the probability that the process

   (a) becomes extinct,

   (b) never reaches State 2,

   (c) never reaches State 11.

5. Customers of a store arrive at an average rate of 4.5 *clusters* per hour. The cluster sizes are independent of each other and have the negative binomial distribution with parameters $p = 0.37$ and $k = 2$ (the smallest cluster size is 2). The time is 8:16. Find the probability that

   (a) the 4^{rd} cluster (from now) will arrive before 8:32,

   (b) more than 13 *customers* will arrive between now and 8:32,

   (c) the next three clusters will have more than 11 customers in total.

6. Assuming

$$\mathbb{A} = \begin{bmatrix} 1 & 6 & 4 & 6 & 10 \\ 0 & -5 & -4 & -4 & -8 \\ -2 & 0 & 2 & 1 & 4 \\ 2 & -4 & -5 & -4 & -8 \\ 0 & 6 & 5 & 5 & 9 \end{bmatrix}$$

   (a) express $\exp(\mathbb{A}\cdot t)$ as a liner combination of the corresponding constituent matrices; in your booklet, refer to them as $\mathbb{C}_1$ to $\mathbb{C}_5$ - spell them out only in Maple (which I need to see as well),

   (b) then do the same with $(3\mathbb{I} + \mathbb{A} \cdot t)^{-3}$.

7. Consider an $M/M/4$ queue with the arrival rate of 13.7 customers per hour and the mean service time of 16 minutes. Compute the long-run

   (a) server utilization factor,

   (b) average waiting time (in minutes and seconds).

   (c) Assuming the process is now in State 6 (two customers waiting), what is the expected time until it enters (for the first time from now) State 3 (one server idling)?

8. Find (without pdsolve) the general solution to

   (a)

$$(z \; \ln z \; \cos z - \sin z)\overset{\bullet}{P}(z,t) + z \; P'(z,t) \sin^2 z = 0$$

   (your answer must be an explicitly *real* function of z and t),

   (b)

$$e^z \sin z \; (\sin z + \cos z)\overset{\bullet}{P}(z,t) = P'(z,t) \sin z + P(z,t)(\sin z + \cos z)$$

   (c) and the particular solution to the previous equation (Part b) which meets

$$P(z,0) = 1$$

   (spell out what your $g(x)$ is as a function of x).

9. Consider a Brownian motion with no drift, $c = 14.7$ m^2 per hour, and the initial value (at $t = 0$) of -3.7 m. Compute the probability that -4 m $< X(2 \text{ hours}) < 0$ m, given that

   (a) $X(1 \text{ hour}) = -2.9$ m,

   (b) $X(1 \text{ hour}) = -2.9$ m and $X(3.5 \text{ hours}) = -1.3$ m,

   (c) there is an absorbing barrier at 0 m (no other information given, beyond the initial value),

   (d) there is an absorbing barrier at -1 m (ditto).

10. Consider a 3D Poisson process with the average density of 'points' being 2.8 per meter^3. Find the *expected*

    (a) distance, from the origin, to the fourth nearest point, and the corresponding standard deviation,

    (b) number of points inside the region defined by

    $$x^2 + y^2 < z^2 \qquad \text{and} \qquad 0 < z < 2$$

    and the corresponding standard deviation (hint: first, identify the *shape* of this region).

    (c) Given that there are exactly 21 points inside the region of Part (b), what is the conditional probability that fewer than 15 of them have $z > 1$?

11. Consider a CTMC with States 1 up to 5 and the following (per hour) transition rates

$$\begin{bmatrix} \times & 2.1 & 3.2 & 1.3 & 0.9 \\ 0.8 & \times & 2.3 & 1.9 & 2.0 \\ 3.7 & 1.1 & \times & 2.0 & 1.7 \\ 4.0 & 0 & 2.7 & \times & 0.6 \\ 0.8 & 1.2 & 1.9 & 2.1 & \times \end{bmatrix}$$

Assuming that the process is currently in State 4, find the

(a) probability of visiting State 1 *before* vising State 5,

(b) expected value and the corresponding standard deviation of the time of the first entry (from now) to State 2,

(c) expected time of the first *re*-entry (from now) to State 4.