BROCK UNIVERSITY

Final Examination: December 2017
Course: MATH 4F85
Date of Examination: Dec. 15, 2017
Time of Examination: 16:00-19:00

Number of Pages: 4
Number of students: 18
Number of Hours: 3
Instructor: J. Vrbik

Open book exam. Use of Maple and a calculator is allowed.
No examination aids other than those specified on the examination scripts are permitted (this regulation does not preclude special arrangements being made for students with disabilities).

Full credit given for 18 (out of 30) correct and complete answers.
Numerical answers must be given to at least 4 significant digits.

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1. Consider a B&D process with rates given by $\lambda_n = 23$ and $\mu_n = 3.9\ n$ (both per hour), having 36 initial members at 9:36. Find

   (a) the probability that, at 9:49, the process will have more than 25 members,

   (b) the expected time of death (use the xx:yy:zz format) of the 30^{th} initial member (still leaving 6 'survivors'), and the corresponding standard deviation (in minutes and seconds),

   (c) the long-run frequency of visits to State 0 (per week).

2. Consider a B&D process with the following (per hour) rates

State	1	2	3	4	5	6	7
λ_n	1.3	3.2	4.1	0.8	1.3	2.2	0
μ_n	0	2.4	1.9	3.1	2.0	3.2	3.0

   now being in State 3. Compute

   (a) the expected time needed to reach State 7 (for the first time from now) and the corresponding standard deviation,

   (b) the long-run frequency of visits to State 7 (per day),

   (c) the probability that, 52 minutes from now, the process will be in State 7.

3. Consider a 2D Poisson process with an average 'dot' density of 18.3 per meter^2 (meters are the units of our coordinates). Find
   (a) the expected distance from the origin to the 5$^{\text{th}}$ nearest dot, and the corresponding standard deviation,
   (b) the probability of getting more than 270 dots inside the triangle with the following vertices: $(-2, 4)$, $(3, -1)$ and $(3, 5)$,
   (c) given that there are exactly 283 dots inside the triangle of Part (b), what is the conditional probability that at least 50 of them have a negative x coordinate.

4. Consider a B&D process with rates given by $\lambda_n = \dfrac{13\ n}{1+n}$ and $\mu_n = \dfrac{13.9\ n^2}{1+n^2}$ (both per hour; note that State 0 is absorbing) and the initial value of 6. Compute
   (a) the probability that (exactly) 3 transitions later, the process is in State 5,
   (b) the expected time till extinction (i.e. reaching State 0),
   (c) the probability of never reaching State 11 (hint: make it also absorbing).

5. Consider Brownian motion with zero drift and the diffusion coefficient equal to 13.5 mm^2 per hour. Find
   (a)
   $$\Pr\left(X(10{:}23) > 3.6\text{ mm} \cap \min_{9:13<t<10:23} X(t) > 1.4 \mid X(9{:}13) = 5.2\text{ mm}\right)$$
   (b)
   $$\Pr\left(X(10{:}23) > 3.6\text{ mm} \mid X(9{:}13) = 5.2\text{ mm} \cap X(9{:}55) = 2.3\text{ mm}\right)$$
   (c)
   $$\Pr\left(X(10{:}23) > 3.6\text{ mm} \mid X(9{:}13) = 5.2\text{ mm} \cap X(10{:}55) = 2.3\text{ mm}\right)$$

6. Individual customers arrive at a looney store at an average rate of 17.2 per hour. The value of each customer's purchase is a random variable having the *modified* geometric distribution with $p = 0.1$ (they are independent of each other). Compute the probability

   (a) of the store getting *exactly* 7 buying customers before the second browser (those who spend \$0) walks in,

   (b) that the customers who arrive during the next half hour will spend more than \$100 in total,

   (c) that the first two customers will spend an equal amount of money (i.e. each spending \$0, or each spending \$1, etc.).

7. Without Maple (and spelling out your g function), find a solution to each of the following PDEs, subject to the corresponding initial condition

   (a)
$$\begin{aligned} \dot{P}(z,t) \cdot \ln z &= P'(z,t) \\ P(z,0) &= z^2 \cdot \left(1 - 2\ln z + \ln^2 z\right) \end{aligned}$$

   (b)
$$\begin{aligned} \dot{P}(z,t) \cdot (1 + \ln z) &= P'(z,t) + \frac{P(z,t)}{z} \\ P(z,0) &= \ln z \end{aligned}$$

   (c)
$$\begin{aligned} \dot{P}(z,t) &= z \cdot P'(z,t) + P(z,t) \\ P(z,0) &= 1 \end{aligned}$$

8. Consider a B&D process with rates given by $\lambda_n = 23 + n$ and $\mu_n = 3.9\ n$ (both per hour), having 36 initial members at 9:36. Find

   (a) the probability that at 9:49 the process will have more than 25 members,

   (b) the probability that the 30^{th} death of an initial member will happen between 9:55 and 10:10,

   (c) the long-run frequency of visits to State 0 (per week).

9. Consider a M/M/5 queue with an average service time of 17 minutes and potential customers arriving at an average rate of 12.7 per hour, but joining the queue only with the probability of $\frac{10}{10+k^2}$, where k is the number of people they find *waiting* for service (note that this could be zero) when they arrive. Find the long-run
   (a) proportion of lost customers,
   (b) average time spent by the actual customers (those who have not balked) in the system (i.e. both waiting and in service).
   (c) server utilization factor.

10. For the following three matrices, find (exactly, i.e. no decimals) $\exp(\mathbb{A} \cdot t)$. Express each solution as a sum of four (two in Part c) terms, using the corresponding constituent matrices; when this solution turns out to be complex, quote it first in its complex form, but then show also its explicitly *real* version.
   (a)
$$\mathbb{A} = \begin{bmatrix} 3 & 4 & -6 & -1 \\ 5 & 1 & -5 & -1 \\ 6 & 4 & -9 & -1 \\ -5 & 1 & 5 & 3 \end{bmatrix}$$
   (b)
$$\mathbb{A} = \begin{bmatrix} -7 & 4 & 0 & -5 \\ -2 & 3 & -1 & -1 \\ 7 & -3 & 1 & 4 \\ 8 & -3 & -1 & 7 \end{bmatrix}$$
   (c)
$$\mathbb{A} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$