BROCK UNIVERSITY

Final Examination: December 2018
Course: MATH 4P85
Date of Examination: Dec. 15, 2018
Time of Examination: 19:00-21:00

Number of Pages: 4
Number of students: 13
Number of Hours: 3
Instructor: J. Vrbik

Open book exam. Use of Maple and a calculator is allowed.
No examination aids other than those specified on the examination scripts are permitted (this regulation does not preclude special arrangements being made for students with disabilities).

Full credit given for 18 (out of 32) correct and complete answers.
Numerical answers must be given to at least 4 significant digits.
All answers must be entered in the examination booklet to get proper credit.
Email your Maple to jvrbik@brocku.ca before handing in your booklet.

1. Consider a M/M/1 queue with the average arrival rate of 8.3 customers per hour and a service time of 10 minutes. A customer who arrives when there are fewer than 5 people waiting for service joins the queue; when there are 5 people waiting, the newcomer must leave, never to come back (this is referred to as a 'finite waiting room' model - a special case of 'balking'). Compute the long-run
   (a) server utilization factor,
   (b) percentage of lost customers,
   (c) frequency of visits to State 0 (per 24 hour day),
   (d) average duration of the server's busy *period*,
   (e) average *waiting* time (of customers who did not have to leave, in minutes).
   (f) What is the probability that the server will be idling 17 minutes from now, given that there are currently 2 customers waiting for service (hint: use the CTMC technique - after all, this process has only 7 possible states).

2. Consider a 3D Poisson process with an average 'dot' density of 7.3 per meter^3 (meters are units of our coordinate system). Compute
   (a) the expected number of dots (and the corresponding standard deviation) inside the region defined by the following two inequalities
   $$x^2 + y^2 + z^2 < 25 \qquad \text{and} \qquad 0 < z$$
   (b) the expected distance from the origin to the 3$^{\text{rd}}$ nearest dot *inside the region* of Part (a).

3. Consider a B&D process with the following (per hour) rates

$$\begin{aligned} \lambda_n &= \frac{3.2n}{2.7+n} \\ \mu_n &= 1.5\ln(1+n) \end{aligned}$$

(note that State 0 is absorbing) being currently in State 3. Find

   (a) the probability of its ultimate absorption in State 0,

   (b) the probability that, three transitions later, the process is in State 4,

   (c) the expected time till absorption in State 0,

   (d) the probability of visiting State 1 more than once (hint: first figure out *how* it can happen),

   (e) the probability that the process will never reach State 6 (hint: make State 6 also absorbing and use the CTMC technique).

4. Consider Brownian motion with *zero* drift and the diffusion coefficient equal to 3.5 inch^2 per hour. Find the probability that

   (a) the process has a value between 2.1 and 4.7 inches at 10:27, given that it was observed to have the value of 3.9 inches at 7:43 *and* the value of 5.1 inches at 9:07,

   (b) the process has a value between 2.1 and 4.7 inches at 10:27, given that it was observed to have the value of 3.9 inches at 11:43 *and* the value of 5.1 inches at 9:07,

   (c) the process has a value between 2.1 and 4.7 inches at 10:27, given that it was observed to have the value of 3.9 inches at 7:43 *and* there is an absorbing barrier at 5.1 inches.

5. Consider an $\mathsf{M/M/\infty}$ queue with the average arrival rate of 14.3 customers per hour and the expected service time of 1 hour 9 minutes and 23 seconds, currently serving 13 customers. Find

   (a) the probability that, 12 minutes later, there are *more than* 15 customers being served,

   (b) the probability that, during the next 12 minutes, exactly 2 of the initial 13 customers will finish their service *while* more than 4 new customers will arrive (hint: due to independance, compute and *multiply* the two answers),

   (c) the long-run proportion of time with more than 15 customers in service.

6. Consider a (non-homogeneous) Poisson process with cutomers arriving at an average (per hour) rate of

$$\lambda(t) = \begin{cases} 7.3 + 1.1t & \text{when} \quad 0 < t < 2 \\ 10.2 - 1.3t & \text{when} \quad 2 \le t \le 5 \\ 2.7 & \text{when} \quad 5 < t < 8 \\ \dfrac{17}{t} & \text{when} \quad t \ge 8 \end{cases}$$

where t is time set to 0 at the store's opening (at 8:00), in hours (note that $t = 2$ thus represents 10:00, etc.). Compute

   (a) the probability that the 25^{th} customer of the day will arrive between 11:07 and 11:43

   (b) the probability of getting between 5 and 10 customers (inclusive) between 11:07 and 11:43,

   (c) the expected time of arrival of the 15^{th} customer (using the xx:yy:zz fomat, i.e. the usual *clock* time), and the corresponding standard deviation (in minutes and seconds).

7. Find $\exp(\mathbb{A} \cdot t)$, where $\mathbb{A}$ is the following matrix

$$\begin{bmatrix} -6 & -2 & \frac{15}{2} & \frac{1}{2} \\ -2 & -5 & 3 & 1 \\ -2 & -2 & 0 & 1 \\ -4 & -6 & 8 & -1 \end{bmatrix}$$

   (a) in its constituent-matrix form, using *complex* terms when necessary.

   (b) Convert the two complex terms of the previous solution into an explicitely real form.

8. Consider the following PDE

$$\dot{P}(z,t) + z^2 P'(z,t) = \tan\left(\frac{1}{z}\right) \cdot P(z,t)$$

Without Maple, find

   (a) its general solution,

   (b) the solution which satisfies

$$P(z,0) = \sin\left(\frac{2}{z}\right)$$

   Make sure to clearly spell out what you are using for the $g(x)$ function of Part (a).

9. Consider a $\mathsf{M/M/6}$ queue with an average service time of 21 minutes and customers arriving at an average rate of 10.7 per hour. Find the long-run

   (a) proportion of time with no lineup (nobody waiting for service),

   (b) average time customers spend in the system (i.e. both waiting and in service), in minutes.

   When Mr. Smith arrives, there are (exactly) 3 people waiting for service. Find the probability that

   (c) (exactly) 2 new customers will arrive before the next departure - hint: consider the next three transitions,

   (d) Mr. Smith's *waiting* time is less than an hour,

   (e) his *service* time is less than 15 minutes,

   (f) more than 5 new customers arrive while he is *waiting* for service (hint: recall Poison process of random duration).