MATH 4P85 FIRST MIDTERM FEBRUARY 9, 2009

One sheet of notes and a Maple workspace (loaded from a memory stick) containing any information are allowed. Full credit given for nine (out of 15) correct and complete solutions. Please, give all answers to four significant digits. **Duration: 50 minutes**

---

1. Consider a three-dimensional Poisson process in the usual x-y-z space, with the average density of 'stars' of 5.36 per light-year^3. Find

   (a) the expected distance from the origin to the nearest star, and the corresponding standard deviation,

   (b) the probability that fewer than 50 stars are inside the region
   $$(x-3)^2 + (y+5)^2 + z^2 \le 3$$

   (c) the expected value and the corresponding standard deviation of the number of stars inside the region defined by
   $$(x-3)^2 + (y+5)^2 \le 3 \quad \text{and} \quad -2 \le z \le 3$$

2. Customers enter a store at a rate of 10.8 per hour. Each of them buys (instantly, we assume) a random number of items chosen (independently of other customers) from the following distribution:

# of items	0	1	2	3
Probability	0.23	0.48	0.21	0.08

   Find:

   (a) the expected value and standard deviation of the total number of items bought between 9:08 and 9:43,

   (b) the probability that more than 12 items are bought between 9:08 and 9:43,

   (c) the probability that *all* customers who arrive between 9:08 and 9:43 buy *at least one* item each. Hint: Remember splitting a Poisson process into two independent Poisson processes ('buyers' and 'non-buyers', in this case).

3. Consider the $M/G/\infty$ queue with $\lambda = 12.3$ per hour, and service times having a distribution with the following probability density function (where s is measured in *minutes*):

$$g(s) = \begin{cases} \frac{s^2}{16}\exp(-s/2) & s > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find

(a) the probability that 8 minutes after opening (with no customers waiting), the number of customers being served is the same as the number of customers who have already left,

(b) the long-run proportion of time with more than 3 customers being served,

(c) the mean and standard deviation of the number of customers who arrive while the third customer is being served. Hint: Remember Poisson process of random duration.

4. Customers enter a store at a (time-dependent) rate of

$$\lambda(t) = \begin{cases} 12 + t & t < 4 \\ 16 - t & 4 \le t < 6 \\ 10 & 6 \le t \end{cases}$$

per hour, where t is time (in hours) since the store's opening at 8:30 (which thus represents $t = 0$). Find

(a) the probability of at least 20 arrivals between 11:50 and 13:15,

(b) the probability that the second customer of the day will arrive between 8:35 and 8:45,

(c) the probability that the first customer arrives before 8:35 while the second one arrives before 8:45 (this is *one* question - we want *both* of these to happen). Hint: Partition the sample space according to the number of arrivals during the first 5 minutes, then use the formula of total probability to find the probability of at least one arrival during the first 5 minutes *and* at least two arrivals during the first 15 minutes.

5. Consider two *independent* random variables X and Y, having the following probability generating function, respectively:

$$
\begin{aligned}
P_x(z) &= \frac{1+2z+3z^2}{6} \\
P_y(z) &= \frac{z}{(3-2z)^4}
\end{aligned}
$$

Find

(a) the mean and variance of $X - Y$,

(b) $\Pr(X + Y > 9)$,

(c) $\Pr(\sum_{i=1}^{Y} X_i > 9)$, where X_1, X_2, $X_3, ..$ are independent random variables, all having the same distribution as X.