MATH 4P85 SECOND MIDTERM NOVEMBER 14, 2019

Open book exam. **Duration: 1 hour**
Maple allowed except in questions with the ■ mark.
All answers must be entered in your booklet.
Full credit given for correctly answering 5 (out of 10) questions.

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1. Consider a LGWI process with the following rates

$$\begin{aligned} \lambda_n &= 23 + 6.1n \text{ per hour} \\ \mu_n &= 6.2n \text{ per hour} \end{aligned}$$

starting with 11 members at 8:17. Compute

(a) ■ the probability that the *last* one of the 11 initial members dies between 8:47 and 9:21,

(b) ■ the expected value and standard deviation of the number of members of the *native* sub-population at 9:03,

(c) the long-run frequency of visits to State 230,

(d) ■ the probability of having between 4 and 7 (inclusive) *surviving* immigrants (ignore their descendents) at 9:03.

2. ■ Find the solution to the following PDE and the corresponding initial condition. Provide details of your computation in your booklet; spell out the exact form of your $g(x)$ as a function of $x$.

(a)

$$\begin{aligned} z \cdot \dot{P}(z,t) &= (1+z^2) \cdot P'(z,t) \\ P(z,0) &= z \end{aligned}$$

(b)

$$\begin{aligned} z \cdot \dot{P}(z,t) &= (1+z^2) \cdot \big(P'(z,t) - P(z,t)\big) \\ P(z,0) &= e^z \end{aligned}$$

3. Consider an M/M/5 queue with the average arrival rate of 5.3 customers per hour, the average service time of 47 minutes, and the initial value of 4 customers (all in service - no one is waiting). What is

   (a) the expected time to reach, for the first time from now, State 0 (with all five servers idling),

   (b) ■ the probability of getting *at least* three arrivals before the first departure (from now).

4. Consider a queueing system with *one* server and the following arrival and departure rates

$$\begin{aligned}\lambda_n &= 1.6 \exp\left(-\tfrac{n}{3}\right)\\ \mu_n &= \frac{1.6n}{n+2.3}\end{aligned}$$

   (note that customers are less likely to join the system when the length of the queue increases, while the server starts working faster). Compute the long-run

   (a) proportion of time the server is busy,

   (b) average length of the actual queue.