1.

(a) Use LGWI model with $\lambda = \frac{63}{24}$ per hour, $\mu = \frac{74}{24}$ per hour, $a = \frac{32}{24}$ hours, $t = \frac{99}{60}$ hours, and $i = 9$. Expanding PGF of $X(t)$, and adding the coefficients of z^4 to z^{10} yields 37.18%.

(b) Using r_t of Part a), with t being variable, the PDF of 'native extinction time' T is

$$f(t) = \frac{dr_t^9}{dt}$$

implying

$$\mu_T = \int_0^\infty t \cdot f(t)dt = 2.9009 \text{ hours}$$

which translates to 11:06:03, and

$$\sigma_T = \sqrt{\int_0^\infty (t-\mu_T)^2 \cdot f(t)dt} = 2.1709 \text{ hours}$$

representing 2 hours, 10 minutes and 15 seconds.

(c) Using the M/M/∞ model with parameters of Part a), except now there is no λ, and $i = 0$, we get 0.4298 and 0.6556

2.

(a) Using the PGF from Question 1a), but setting $i = 0$, we get 1.543 and 2.497 for the mean and standard deviation, respectively, of surviving immigrants and their descendents.

(b) Evaluating the stationary PGF of the M/M/∞ distribution at $z = 0$, we get $e^{-a/\mu} = 64.89\%$.

(c) The number of immigrants has the Poisson distribution with $\Lambda = \lambda \cdot t = 4.33125$; the probability that an immigrant has no offspring is $p_c = \frac{\mu}{\lambda+\mu} = \frac{74}{137}$. Using the formula of total probability, we get

$$\sum_{j=0}^{\infty} \frac{\Lambda^j}{j!} e^{-\Lambda} \cdot p_c^j = e^{\Lambda(p_c-1)} = 13.65\%$$

3. Since $a(z) = \frac{1}{1+\ln z}$, we get for the general solution

(a)

$$g\left(t + \int (1+\ln z)dz\right) = g\left(t + z\ln z\right)$$

(b) We need $g\left(z \ln z\right) = z^z$, which is clearly achieved by $g(x) = e^x$ (note that $e^{z\ln z} = z^z$). The particular solution is thus

$$e^t \cdot z^z$$

4. Since $a(z) = \frac{\sin z}{\cos z}$ and $b(z) = 1$, we get for the general solution

   (a)
   $$g\left(t + \int \frac{\cos z}{\sin z} dz\right) \cdot \exp\left(-\int \frac{\cos z}{\sin z} dz\right) = g\left(t + \ln(\sin z)\right) \cdot \exp\left(-\ln(\sin z)\right) = \frac{g\left(t + \ln(\sin z)\right)}{\sin z}$$

   (b) We need
   $$\frac{g(\ln(\sin(z))}{\sin z} = 1$$
   or
   $$g(\ln(\sin(z)) = \sin z$$
   This is achieved by $g(x) = e^x$ again. The particular solution is thus
   $$\frac{\exp\left(t + \ln(\sin z)\right)}{\sin z} = \frac{e^t \cdot \sin z}{\sin z} = e^t$$

5. This is power-supply model with $\lambda = 0.6$ per minute, $\mu = 0.9$ per minute, $N = 12$ and $i = 6$.

   (a)
   $$\frac{\lambda_6}{\lambda_6 + \mu_6} \cdot \frac{\lambda_7}{\lambda_7 + \mu_7} \cdot \frac{\lambda_8}{\lambda_8 + \mu_8} =$$
   $$\frac{7.2 - 0.6 \times 6}{7.2 - 0.6 \times 6 + 0.9 \times 6} \cdot \frac{7.2 - 0.6 \times 7}{7.2 - 0.6 \times 7 + 0.9 \times 7} \cdot \frac{7.2 - 0.6 \times 8}{7.2 - 0.6 \times 8 + 0.9 \times 8} = 3.226\%$$

   (b) Based on the PGF of the process (or the corresponding formulas), and using $t = 2 + \frac{54}{60}$ minutes, we get 4.815 and 1.698 for the mean and standard deviation, respectively, of $X(t)$.

   (c) Using $p_\infty = \frac{\lambda}{\lambda+\mu} = 0.4$ as the probability of 'success' of the stationary $\mathcal{B}(12, p_\infty)$ distribution, we get, for the long-run frequency of visits to State 12
   $$p_{12} \cdot (\lambda_{12} + \mu_{12}) = p_\infty^{12} \cdot 0.9 \times 12 \text{ per minute}$$
   or
   $$0.4^{12} \times 0.9 \times 12 \times 60 = 0.01087 \text{ per hour}$$
   The expected duration of each visit is
   $$\frac{1}{\lambda_{12} + \mu_{12}} = \frac{1}{0.9 \times 12} \text{ minutes}$$
   or
   $$\frac{60}{0.9 \times 12} = 5.556 \text{ seconds}$$