1.

(a)

$$p_t = e^{-2.43\times 32/60}$$

$$p_t^3 \sum_{j=3}^{11} \tbinom{j-1}{2} (1-p_t)^{j-3} = 61.54\%$$

(b) Now

$$p_t = e^{-2.43\times(15/60+37/3600)}$$

which results in the following expected value

$$\frac{3}{p_t} = 5.647$$

and standard deviation of

$$\sqrt{\frac{3}{p_t} \times \left(\frac{1}{p_t} - 1\right)} = 2.232$$

2.

(a) Since

$$p_t = e^{-2.43\times 32/60}$$

one gets

$$\sum_{i=0}^{11} \binom{31}{i} p_t^i \times (1-p_t)^{31-i} = 88.59\%$$

(b) Since

$$p_t = e^{-2.43\times(15/60+37/3600)}$$

the mean and standard deviation are

$$\begin{aligned} 31 \times p_t &= 16.47 \\ \sqrt{31 \times p_t \times (1-p_t)} &= 2.778 \end{aligned}$$

respectively.

(c) This equals the probability that it takes less than 2 hours to get extinct, minus the probability that it takes less than one hour, namely

$$\left(1 - e^{-2.43\times 2}\right)^{31} - \left(1 - e^{-2.43}\right)^{31} = 72.82\%$$

(d)

$$\begin{aligned} \tfrac{1}{2.43} \sum_{i=1}^{31} \tfrac{1}{i} &= 1.657 \quad \text{hours} \\ \tfrac{1}{2.43} \sqrt{\sum_{i=1}^{31} \tfrac{1}{i^2}} &= 0.5227 \quad \text{hours} \end{aligned}$$

(could be converted to minutes and seconds).

3.

$$\begin{aligned} p_t &= \frac{(3.1-4.1)e^{-(3.1-4.1)\times 2/3}}{3.1-4.1e^{-(3.1-4.1)\times 2/3}} \\ r_t &= \frac{4.1(1-e^{-(3.1-4.1)\times 2/3})}{3.1-4.1e^{-(3-4)\times 2/3}} \end{aligned}$$

(a)

$$1 - r_t^4 - \sum_{j=1}^{4} \binom{4}{j}(1-r_t)^j r_t^{4-j} \cdot p_t^j \sum_{k=j}^{4} \binom{k-1}{j-1}(1-p_t)^{k-j} = 15.39\%$$

Student should get this answer by expanding

$$\left(r_t + (1-r_t) \times \frac{p_t\ z}{1-(1-p_t)z}\right)^4$$

(b) The corresponding expected value and standard deviation are

$$\begin{aligned} \frac{4 \times (1-r_t)}{p_t} &= 2.054 \\ \sqrt{4 \times \frac{(1-r_t)(r_t+1-p_t)}{p_t^2}} &= 2.682 \end{aligned}$$

respectively.

(c) We have to recompute

$$r_t = \frac{4.1(1-e^{-(3.1-4.1)\times 23/60})}{3.1-4.1e^{-(3.1-4.1)\times 23/60}}$$

Answer: $r_t^4 = 18.63\%$.

(d)

$$\begin{aligned} r_t &= \frac{4.1(1-e^{-(3.1-4.1)t})}{3.1-4.1e^{-(3.1-4.1)t}} \\ \int_0^\infty t \cdot \frac{d}{dt}(r_t^4)dt &= 1.1107 \quad \text{hours} \\ \sqrt{\int_0^\infty (t-1.1107)^2 \cdot \frac{d}{dt}(r_t^4)dt} &= 0.9461 \quad \text{hours} \end{aligned}$$

4.

(a)

$$\begin{aligned} p_t &= \frac{(4.1-3.1)e^{-(4.1-3.1)\times 2/3}}{4.1-3.1e^{-(4.1-3.1)\times 2/3}} \\ r_t &= \frac{3.1(1-e^{-(4.1-3.1)\times 2/3})}{4.1-3.1e^{-(4.1-3.1)\times 2/3}} \end{aligned}$$

$$1 - r_t^4 - \sum_{j=1}^{4} \binom{4}{j} (1-r_t)^j r_t^{4-j} \cdot p_t^j \sum_{k=j}^{4} \binom{k-1}{j-1} (1-p_t)^{k-j} = 59.19\%$$

They should get it by expanding

$$\left(r_t + (1-r_t) \times \frac{p_t \ z}{1-(1-p_t)z} \right)^4$$

(b) The expected value and standard deviation of $X(33 \text{ min})$

$$\begin{aligned} p_t &= \frac{(4.1-3.1)e^{-(4.1-3.1)\times 33/60}}{4.1-3.1e^{-(4.1-3.1)\times 33/60}} \\ r_t &= \frac{3.1(1-e^{-(4.1-3.1)\times 33/60})}{4.1-3.1e^{-(4.1-3.1)\times 33/60}} \end{aligned}$$

$$\begin{aligned} \frac{4 \times (1-r_t)}{p_t} &= 6.933 \\ \sqrt{4 \times \frac{(1-r_t)(r_t+1-p_t)}{p_t^2}} &= 6.050 \end{aligned}$$

(one could also extract these from the PGF).

(c) We have to recompute

$$r_t = \frac{3.1(1-e^{-(4.1-3.1)\times 23/60})}{4.1-3.1e^{-(4.1-3.1)\times 23/60}}$$

$r_t^4 = 6.089\%$.

(d)

$$\left(\tfrac{3.1}{4.1}\right)^4 = 32.68\%$$

5.

(a) Since

$$\int z e^{-z} dz = -(1+z)e^{-z}$$

we get

$$g[t + (1+z)e^{-z}]$$

(b) We need

$$g\left[(1+z)e^{-z}\right] = \ln(1+z) - z$$

which is clearly met by $g(x) = \ln(x)$. Thus, the answer is

$$\ln\left[(1+z)e^{-z} + t\right]$$