We know (from *Fourier-series* theory) that

$$\rho_k = \frac{1}{2\pi}\int_0^{2\pi} \omega(\beta)\cos(k\beta)d\beta$$

Introducing $y \equiv \exp(i\beta)$, which implies that

$$\cos(k\beta) = \frac{y^k + y^{-k}}{2} \qquad \text{and} \qquad d\beta = \frac{dy}{iy}$$

this can be converted to

$$\frac{1}{2\pi i}\oint \omega(y)\left(\frac{y^{k-1}+y^{-k-1}}{2}\right)dy \tag{1}$$

where the integration is over the *unit circle* centered at 0 (counterclockwise). The theory of *contour integration* tells us that the result is the sum of *residues* of the integrand at all *singularities* inside this circle. Note that the ω (being, originally, a function of

$$\cos\beta = \frac{y+\frac{1}{y}}{2}$$

has the $\omega(y) = \omega(\frac{1}{y})$ symmetry.

Expanding $\omega(y)$, which is a *rational function* of y, in terms of *partial fractions* makes it quite easy to identify the singularities, and to find their residues. One should realize that each term of this expansion will have the form of either

$$c \cdot y^m \tag{2}$$

where c is a constant (different for different terms) and m is an integer (positive, negative, or zero), or of

$$\frac{c}{(y-\lambda)^\ell} \tag{3}$$

where c and $\lambda \neq 0$ are two constants (potentially complex), and ℓ is a positive integer (usually equal to 1). Clearly any such term with complex c and λ will have its *complex conjugate* counterpart. But, more importantly, due to the $y \leftrightarrow \frac{1}{y}$ symmetry, each term of the type $c \cdot y^m$ must be 'paired' with the corresponding $c \cdot y^{-m}$ term (this time, with the *same* c), and similarly every term of type (3) - regardless whether c and λ are complex or real - will have its 'dual' *set* of terms, given by the partial-fraction expansion of

$$\frac{c}{\left(\frac{1}{y}-\lambda\right)^\ell} = \frac{cy^\ell}{(1-y\lambda)^\ell} = \frac{c\left(-\frac{1}{\lambda}\right)^\ell y^\ell}{\left(y-\frac{1}{\lambda}\right)^\ell} \tag{4}$$

namely

$$c\left(-\frac{1}{\lambda}\right)^\ell \sum_{j=0}^{\ell}\binom{\ell}{j}\frac{\lambda^{-j}}{\left(y-\frac{1}{\lambda}\right)^j}$$

(the details of which become irrelevant shortly). Note that the absolute value of each λ must be different from 1 (to represent a stationary model).

One can prove that the contributions of cy^m and cy^{-m} to (1) are identical (for all positive-integer values of k); the same is true for the (3) and (4) pair (see below). This means that we can divide terms of the partial-fraction expansion of $\omega(y)$ into two parts:

1. those which have the form of either (2) with m *negative* or of (3) with $|\lambda| < 1$,
2. all their *dual* terms - these consist of positive powers of y, and terms or Type (3) with $|\lambda| > 1$, all easily identifiable.

The constant term is an exception to this, but since it contributes only to $\rho_0 \equiv 1$, we can ignore it. We have excluded the $|\lambda| = 1$ possibility which cannot happen, as explained earlier.

All we have to do in the end is to find the sum of residues of terms of Type 1 and multiply the result by 2 (totally ignoring terms of Type 2, which would contribute the same amount - that means we never need to utilize the last two formulas). Note that multiplying by 2 is the same as replacing

$$\left(\frac{y^{k-1} + y^{-k-1}}{2}\right)$$

in (1) by $\left(y^{k-1} + y^{-k-1}\right)$ - that's what we will do from now on.

Proof. First we prove that

$$\frac{1}{2\pi i} \oint y^m \left(y^{k-1} + y^{-k-1}\right) dy = \frac{1}{2\pi i} \oint y^{-m} \left(y^{k-1} + y^{-k-1}\right) dy = \delta_{m,k}$$

when m is a positive integer. Note that $\delta_{m,k}$ is the Kronecker's delta (equal to 1 when $m = k$, equal to 0 otherwise).

The result is quite obvious by realizing that the only power of y which integrates (in the above sense) to $2\pi i$ is y^{-1} - all the other integer powers yield zero. It is thus the y^{m-k-1} part of the first integral and the y^{-m+k-1} part of the second integral which result in $\delta_{m,k}$ (the y^{m+k-1} and y^{-m-k-1} terms contribute 0).

Secondly, we have to deal with the

$$\frac{1}{2\pi i} \oint (y - \lambda)^{-\ell} \left(y^{k-1} + y^{-k-1}\right) dy = \frac{1}{2\pi i} \oint \frac{\left(-\frac{1}{\lambda}\right)^{\ell} y^{\ell}}{\left(y - \frac{1}{\lambda}\right)^{\ell}} \left(y^{k-1} + y^{-k-1}\right) dy$$

identity, assuming that $|\lambda| = 1$. The LHS integral has two singularities inside the unit circle: one at 0 and the other at λ. The first one contributes (another well-known result of contour integration):

$$\frac{1}{k!} \cdot \left.\frac{d^k (y - \lambda)^{-\ell}}{dy^k}\right|_{y=0} = \frac{(-\ell)(-\ell - 1)...(-\ell - k + 1)}{k!} \cdot (-\lambda)^{-\ell-k} = (-1)^{\ell} \binom{\ell + k - 1}{k} \lambda^{-\ell-k}$$

the second one adds

$$\frac{1}{(\ell-1)!}\cdot\left.\frac{d^k y^{k-1}}{dy^{\ell-1}}\right|_{y=\lambda}+\frac{1}{(\ell-1)!}\cdot\left.\frac{d^k y^{-k-1}}{dy^{\ell-1}}\right|_{y=\lambda}=$$
$$\frac{(k-1)(k-2)...(k-\ell+1)}{(\ell-1)!}\cdot\lambda^{k-\ell}+\frac{(-k-1)(-k-2)...(-k-\ell+1)}{(\ell-1)!}\cdot\lambda^{-k-\ell}=$$
$$\binom{k-1}{\ell-1}\lambda^{k-\ell}+(-1)^{\ell-1}\binom{\ell+k-1}{\ell-1}\lambda^{-\ell-k}$$

Clearly, the second term cancels the contribution of the 0 singularity, with the net result of

$$\frac{1}{2\pi i}\oint(y-\lambda)^{-\ell}\left(y^{k-1}+y^{-k-1}\right)dy=\binom{k-1}{\ell-1}\lambda^{k-\ell} \tag{5}$$

The RHS integral has only the 0 singularity contributing (the $y=\frac{1}{\lambda}$ singularity is outside the unit circle), yielding

$$\left(-\frac{1}{\lambda}\right)^{\ell}\cdot\frac{1}{(k-\ell)!}\cdot\left.\frac{d^k(y-\frac{1}{\lambda})^{-\ell}}{dy^{k-\ell}}\right|_{y=0}=\left(-\frac{1}{\lambda}\right)^{\ell}\cdot\frac{(-\ell)(-\ell-1)...(-k+1)}{(k-\ell)!}\cdot\left(-\frac{1}{\lambda}\right)^{-k}=\binom{k-1}{k-\ell}\lambda^{k-\ell}$$

which is the same as (5).

We should mention that in the most common case of $\ell=1$, this answer boils down to λ^{k-1} (for $\ell=2$ we are getting $(k-1)\lambda^{k-2}$, for $\ell=3$ this becomes $\frac{(k-1)(k-2)}{2}\lambda^{k-3}$, etc.). ∎

Example: Suppose

$$\omega(\beta)=\frac{12+6\cos(\beta)-6\cos(2\beta)-12\cos(3\beta)}{17+8\cos(2\beta)}$$

This is equivalent to

$$\omega(y)=-\frac{3}{2}-\frac{3}{4}\left(y+\frac{1}{y}\right)+\frac{\frac{15}{32}+\frac{5i}{8}}{y-\frac{i}{2}}+\frac{\frac{15}{32}-\frac{5i}{8}}{y+\frac{i}{2}}+\frac{\frac{15}{8}-\frac{5i}{2}}{y-2i}+\frac{\frac{15}{8}+\frac{5i}{2}}{y+2i}$$

where $y=\exp(i\beta)$. We thus get (almost immediately)

$$\rho_k=\left(\frac{15}{32}+\frac{5i}{8}\right)\left(\frac{i}{2}\right)^{k-1}+\left(\frac{15}{32}-\frac{5i}{8}\right)\left(-\frac{i}{2}\right)^{k-1} \qquad \text{for } k>1$$

and

$$\rho_1=\left(\frac{15}{32}+\frac{5i}{8}\right)+\left(\frac{15}{32}-\frac{5i}{8}\right)-\frac{3}{4}=\frac{3}{16}$$