Distribution of the circular sample correlation coefficient for Markov model

Circular Markov model:

$$X_i = \rho X_{i-1} + \varepsilon_i$$

for $i = 1, 2, ...n$, but also $X_n \equiv X_0$. This can be achieved by first starting with $X_0 = 0$ and then, in a second pass (using the same ε values), setting it to the value of X_n from the first pass. It is easy to see that the corresponding pdf is

$$\begin{aligned}
& \frac{1-\rho^n}{(2\pi)^{n/2}\sigma^n} \exp\left(-\frac{\sum_{i=1}^n (X_i - \rho X_{i-1})^2}{2\sigma^2}\right) \\
& \frac{1-\rho^n}{(2\pi)^{n/2}\sigma^n} \exp\left(-\frac{\sum_{i=1}^n (1+\rho^2)X_i^2 - 2\rho\sum_{i=1}^n X_{i-1}X_i}{2\sigma^2}\right) \\
= \; & \frac{1-\rho^n}{(2\pi)^{n/2}\sigma^n} \exp\left(-\frac{\mathbf{X}^{\mathrm{T}}\mathbb{A}\mathbf{X}}{2\sigma^2}\right)
\end{aligned}$$

(with the understanding that $X_0 = X_n$), where

$$\mathbb{A} = \begin{bmatrix} 1+\rho^2 & -\rho & 0 & 0 & \dots & -\rho \\ -\rho & 1+\rho^2 & -\rho & 0 & \dots & 0 \\ 0 & -\rho & 1+\rho^2 & \ddots & \dots & 0 \\ 0 & 0 & -\rho & \ddots & -\rho & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1+\rho^2 & -\rho \\ -\rho & 0 & 0 & \dots & -\rho & 1+\rho^2 \end{bmatrix}$$

(the corresponding inverse is not so simple any more), and $1-\rho^n$ is the Jacobian of the $\varepsilon \to X$ transformation, namely

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & -\rho \\ -\rho & 1 & 0 & 0 & \dots & 0 \\ 0 & -\rho & 1 & \ddots & \dots & 0 \\ 0 & 0 & -\rho & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \dots & -\rho & 1 \end{bmatrix} \mathbf{X}$$

Now, we define a circular serial correlation coefficient by

$$r = \frac{\sum_{i=1}^n X_i X_{i-1}}{\sum_{i=1}^n X_i^2} \equiv \frac{U}{V}$$

Its quite obvious that the distribution of r will be the same for any σ. We will thus set $\sigma = 1$ from now on.

The joint MGF of V and U is, say $M(t_1,t_2)$, equals

$$\frac{1-\rho^n}{(2\pi)^{n/2}}\int\cdots\int\exp\left(-\frac{\mathbf{X}^{\mathrm{T}}\mathbb{A}\mathbf{X}}{2}\right)\exp(t_1\sum_{i=1}^{n}X_i^2+t_2\sum_{i=1}^{n}X_iX_{i-1})dX_1...dX_n$$
$$=\frac{1-\rho^n}{(2\pi)^{n/2}}\int\cdots\int\exp\left(-\frac{\mathbf{X}^{\mathrm{T}}\mathbb{A}_1\mathbf{X}}{2}\right)dX_1...dX_n=\frac{1-\rho^n}{\sqrt{|\mathbb{A}_1|}}$$

where

$$\mathbb{A}_1=\begin{bmatrix} 1+\rho^2-2t_1 & -\rho-t_2 & 0 & 0 & \cdots & -\rho-t_2 \\ -\rho-t_2 & 1+\rho^2-2t_1 & -\rho-t_2 & 0 & \cdots & 0 \\ 0 & -\rho-t_2 & 1+\rho^2-2t_1 & \ddots & \cdots & 0 \\ 0 & 0 & -\rho-t_2 & \ddots & -\rho-t_2 & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1+\rho^2-2t_1 & -\rho-t_2 \\ -\rho-t_2 & 0 & 0 & \cdots & -\rho-t_2 & 1+\rho^2-2t_1 \end{bmatrix}$$

We note that this is a circular matrix, which in general has the form of

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ a_n & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & a_2 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_n & a_1 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_n & a_1 \end{matrix}$$

Its easy to see that

$$\lambda_j=a_1+a_2\omega_j+a_3\omega_j^2+...+a_n\omega_j^{n-1}$$

are n distinct eigenvalues of the previous matrix, where

$$\omega_j=\cos\frac{2\pi j}{n}+i\sin\frac{2\pi j}{n}$$

$j=1,2,...n$ are the n distinct, complex values of $\sqrt[n]{1}$.

Proof: Subtract λ_j from the main diagonal and multiply the resulting columns by 1, ω_j, ω_j^2, ...ω_j^{n-1}, respectively. One can then see that elements of each row add up to 0.

The determinant is just the product of all eigenvalues. For our matrix $\mathbb{A}_1$ this means

$$\prod_{j=1}^{n}\left[1+\rho^2-2t_1-(\rho+t_2)(\omega_j+\omega_j^{-1})\right]=$$
$$(\rho+t_2)^n\prod_{j=1}^{n}\left[\frac{1+\rho^2-2t_1}{\rho+t_2}-(\omega_j+\omega_j^{-1})\right]$$

To simplify this is a bit tricky, we introduce a new variable z by

$$z + \frac{1}{z} = \frac{1+\rho^2 - 2t_1}{\rho + t_2}$$

so that the previous expression becomes

$$\left(\frac{\rho + t_2}{z}\right)^n \prod_{j=1}^{n} \left[z^2 - z(\omega_j + \omega_j^{-1}) + 1\right] =$$
$$\left(\frac{\rho + t_2}{z}\right)^n \prod_{j=1}^{n} (\omega_j - z)(w_j^{-1} - z) =$$
$$\left(\frac{\rho + t_2}{z}\right)^n (1 - z^n)^2$$

where we are taking $|z| < 1$. This is achieved by $z = a - \sqrt{a^2 - 1}$, where $a = \frac{1+\rho^2-2t_1}{2(\rho+t_2)}$.

We thus have

$$M(t_1, t_2) = \frac{1-\rho^n}{1-z^n} \cdot \frac{z^{n/2}}{(\rho + t_2)^{n/2}}$$

Converting to pdf of r

To find the joint pdf of V and U requires the (double) Fourier inverse of $M(t_1, t_2)$, namely

$$f(v, u) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi(t_1, t_2) \exp(-it_1 v - it_2 u) dt_1 dt_2$$

A simple transformation yields the joint pdf of V and $R \equiv \frac{U}{V}$, namely $v \cdot f(v, vr)$. Integrating V out results in the pdf of R (our final result), thus

$$\begin{aligned}
f(r) &= \int_0^{\infty} v \cdot f(v, vr) dv = \int_{-\infty}^{\infty} v \cdot f(v, vr) dv \\
&= \frac{1}{i} \frac{d}{d\theta} \int_{-\infty}^{\infty} e^{iv\theta} f(v, vr) dv \Big|_{\theta=0} \\
&= \frac{\frac{d}{d\theta}}{i(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi(t_1, t_2) \exp(-it_1 v - it_2 vr + iv\theta) dv dt_1 dt_2 \Big|_{\theta=0} \\
&= \frac{\frac{d}{d\theta}}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi(t_1, t_2) \delta(t_1 + t_2 r - \theta) dt_1 dt_2 \Big|_{\theta=0} \\
&= \frac{\frac{d}{d\theta}}{2\pi i} \int_{-\infty}^{\infty} \chi(\theta - t_2 r, t_2) dt_2 \Big|_{\theta=0}
\end{aligned}$$

We will now trade the dt_2 integration for dz. The original definition of z now reads:

$$z + \frac{1}{z} = \frac{1 + \rho^2 - 2i\theta + 2irt_2}{\rho + it_2}$$

Solving for it_2 yields

$$it_2 = -\rho + \frac{z(1 - 2\rho r + \rho^2 - 2i\theta)}{1 - 2rz + z^2}$$

Substituting into $\chi(\theta - rt_2, t_2)$ and multiplying by

$$\frac{dt_2}{dz} = \frac{(1 - z^2)(1 - 2\rho r + \rho^2 - 2i\theta)}{i(1 - 2rz + z^2)^2}$$

yields

$$\frac{1 - \rho^n}{1 - z^n} \cdot \frac{(1 - z^2)(1 - 2rz + z^2)^{n/2-2}}{i(1 - 2\rho r + \rho^2 - 2i\theta)^{n/2-1}}$$

This expression needs to be integrated over the corresponding path (in z, which starts at $r - i\sqrt{1 - r^2}$ and end at $r + i\sqrt{1 - r^2}$), and differentiated with respect to θ. We know that we can modify the path arbitrarily (as long as we don't cross any singularity, which are all on the unit circle), so why not take the straight line connecting the two end points (this also makes z independent of θ)! As a result of all this, we get

$$f(r) = \frac{1}{2\pi i} \frac{(n-2)(1 - \rho^n)}{(1 - 2\rho r + \rho^2)^{n/2}} \int_{r-i\sqrt{1-r^2}}^{r+i\sqrt{1-r^2}} \frac{(1 - z^2)(1 - 2rz + z^2)^{n/2-2}}{1 - z^n} dz$$

When n is even, the last integral (divided by i) can be found analytically to have the value of

$$\frac{1}{n} \sum_{j=1}^{n} \mathrm{Im} \left\{ \left[\omega_j(\omega_j^2 - 1)(1 - 2r\omega_j + \omega_j^2)^{n/2-2} \right] \left[2\ln(r - \cos\varphi_j) - \ln\left(1 - r\cos\varphi_j + \sqrt{1 - r^2}\sin\varphi_j\right) \right] \right\}$$

where $\varphi_j = \frac{2\pi j}{n}$ and $\omega_j = \cos\varphi_j + i\sin\varphi_j$. Watch out, this computation is ill conditioned.