1. Let X_1, X_2, X_3 and X_4 have a multivariate Normal distribution with respective means of 3, -4, 2, 0 and 5, and the variance-covariance matrix of

$$\mathbb{V} = \begin{bmatrix} 11 & -4 & 3 & -2 & 7 \\ -4 & 5 & -2 & 1 & -6 \\ 3 & -2 & 9 & 0 & 10 \\ -2 & 1 & 0 & 1 & -2 \\ 7 & -6 & 10 & -2 & 17 \end{bmatrix}$$

(verify that it is positive definite). Find a 5×5 upper-triangular matrix $\mathbb{A}$ (with positive main-diagonal elements) such that $\mathbb{V} = \mathbb{A}^{\mathrm{T}}\mathbb{A}$. Also: what is the conditional distribution of X_2, X_4 given that $X_1 = 1.5$, $X_3 = -1.2$ and $X_5 = 7$.

2. Find the probability density function of n consecutive observations of a stationary time series based on the Yule model (with 3 parameters: α_1, α_2 and σ^2). Assume that $n \geq 4$.

3. Using the first set of data provided on the web site, find the maximum-likelihood estimators for μ, σ and ρ of the following model:

$$X_i = \mu + \rho(X_{i-1} - \mu) + \varepsilon_i$$

where ε_i are IID from $\mathcal{N}(0, \sigma)$.

4. Using the second set of data and the following formula

$$r_k = \frac{\sum_{i=1}^{N-k} X_i X_{i+k}}{\sum_{i=1}^{N} X_i^2}$$

estimate and plot the values of the first 50 serial correlation coefficients (ρ_1 to ρ_{50}). Also estimate and plot the spectral density (at $\beta = \frac{j \cdot \pi}{40}$ where $j = 0$, 1, ... 40). Smooth out the resulting graph by means of the following 'Tukey' window:

$$\lambda(k) = \frac{1 + \cos\frac{k}{M}}{2}$$

using $M = 25$.

5. Consider the following ARMA model:

$$X_i = -1.6X_{i-1} - 0.9X_{i-2} + \varepsilon_i + \varepsilon_{i-1} + \varepsilon_{i-2}$$

where ε_i are independent, Normally distributed with the mean equal to zero and $\sigma = 1.7$.

   (a) Compute and plot ρ_k for $k = 1, 2, ...50$.

   (b) Find (express as a rational function of $\cos\beta$ and $\cos 2\beta$) and plot the process' spectral density.

(c) Find

$$\Pr(X_{312} > 4.9 \mid X_{311} = -3.5 \cap X_{310} = 2.4 \cap X_{309} = -1.8)$$

6. Suppose we need to estimate the first few moments of

$$g(\bar{X})$$

We already know that

$$\begin{aligned} \mathbb{E}\left[g(\bar{X})\right] &\simeq g(\mu) + g''(\mu)\frac{\sigma^2}{2N} \\ \mathrm{Var}\left[g(\bar{X})\right] &\simeq g'(\mu)^2\frac{\sigma^2}{N} \end{aligned}$$

where μ and σ are the mean and standard deviation of the X distribution, and N is the sample size.

Apply these formulas to find the corresponding approximation to the expected value and variance of $\frac{1}{\bar{X}}$, when sampling from an exponential distribution with the mean equal to β. Compare with the exact results (which, for this simple situation, can be found analytically).

Optional: Find a similar approximation for the third central moment of $g(\bar{X})$.