1. Consider the following ARMA(2,2) model:

$$X_i = 1.6X_{i-1} - 0.9X_{i-2} + \varepsilon_i - 1.8\varepsilon_{i-1} + \varepsilon_{i-2}$$

where ε_i are independent, Normally distributed with the mean equal to zero and $\sigma = 1.9$.

   (a) Compute and plot ρ_k for $k = 1, 2, ...50$.

   (b) Find (express as a rational function of $\cos\beta$ and $\cos 2\beta$) and plot the process' spectral density.

2. Assume the following AR(3) model

$$X_i = 0.9X_{i-1} + 0.81X_{i-2} - 0.729X_{i-3} + \varepsilon_i$$

where ε_i are IID from $\mathcal{N}(0,\ 0.72)$.

   (a) Verify that the process is stationary.

   (b) Compute

$$\Pr(X_{313} > -5 \mid X_{311} = -3.2 \cap X_{310} = 2.1 \cap X_{309} = -1.5 \cap X_{307} = 2.4)$$

   (c) Find $\mathbb{E}\left(X_{200}X_{202}^2X_{205}\right)$.

3. Using Tukey's window with $M = 5,\ 10,\ ...25$ estimate the power spectrum of the given sequence of X_i values (present a formula and the corresponding plot). Which of these do you think best represents the true spectrum? Using the result with $M = 10$, predict (and plot) the power spectrum of a new sequence (shorter by three terms), created by

$$Y_i = X_i - 0.75X_{i-1} + 0.5X_{i-2} + 0.25X_{i-3}$$

Verify your prediction by finding (and plotting) the empirical power spectrum of the new sequence (using the same Tukey's window). Each power-spectrum function should be expanded in terms of $\cos\beta$, $\cos 2\beta$, $\cos 3\beta, ...$

4. Using the set of data provided, find the ML estimators for μ, σ and ρ of the following model:

$$X_i = \mu + \rho(X_{i-1} - \mu) + \varepsilon_i$$

where ε_i are IID from $\mathcal{N}(0, \sigma)$, and $X_0 \equiv X_{100}$ (*circular* condition).

5. Consider the following MA(2) model:

$$X_i = \varepsilon_i + \gamma_1\varepsilon_{i-1} + \gamma_2\varepsilon_{i-2}$$

where ε_i IID from $\mathcal{N}(0, \sigma)$.

(a) Using the data provided, compute the following large-sample estimates of $\mathrm{Var}(X)$, ρ_1 and ρ_2:

$$\frac{\sum_{i=1}^{n} X_i^2}{n}$$
$$\frac{\sum_{i=2}^{n} X_i X_{i-1}}{\sum_{i=1}^{n} X_i^2}$$
$$\frac{\sum_{i=3}^{n} X_i X_{i-2}}{\sum_{i=1}^{n} X_i^2}$$

respectively.

(b) Convert these into the corresponding estimates of γ_1, γ_2 and σ. Why is it possible to get two different answers (find both).

(c) Based on these estimates, find (and plot) the corresponding expression for the (theoretical) spectral density function (are you getting two different answers now)?