n dimensional sphere

of radius r is defined as the following set

$$x_1^2 + x_2^2 + ... + x_n^2 \le r^2$$

Its volume $V_n(r) = r^n V_n(1)$ and surface area $S_n(r)$ are clearly related by

$$S_n(r) = \frac{dV_n(r)}{dr} = nr^{n-1}V_n(1) = r^{n-1}S_n(1)$$

It is easier to find S_n from

$$\pi^{n/2} = \int \cdots \int \exp(-x_1^2 - x_2^2 - ... - x_n^2)dx_1 dx_2 ... dx_n =$$

$$\begin{aligned} S_n(1)\int_0^\infty r^{n-1}\exp(-r^2)dr &= \\ \frac{S_n(1)}{2}\int_0^\infty y^{n/2-1}\exp(-y)dy &= \frac{S_n(1)}{2}\Gamma\left(\frac{n}{2}\right) \end{aligned}$$

which implies that

$$S_n(1) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$$

$$S_n(r) = \frac{2\pi^{n/2}r^{n-1}}{\Gamma\left(\frac{n}{2}\right)}$$

and

$$V_n(r) = \frac{2\pi^{n/2}r^n}{n\Gamma\left(\frac{n}{2}\right)}$$

Sample from standardized bi-variate Normal distribution

has a PDF given by

$$(2\pi)^{-n}(1-\rho^2)^{-n/2}\exp\left(-\frac{\sum(x_i^2 + y_i^2 - 2\rho x_i y_i)}{2(1-\rho^2)}\right)dx_1...dy_n =$$

$$(2\pi)^{-n}(1-\rho^2)^{-n/2} \cdot$$

$$\exp\left(-\frac{n(s_1^2 + \bar{x}^2 + s_2^2 + \bar{y}^2 - 2\rho r s_1 s_2 - 2\rho\bar{x}\bar{y})}{2(1-\rho^2)}\right)dx_1...dy_n$$

where

$$\begin{aligned} \bar{x} &= \frac{\sum x_i}{n} \quad \bar{y} = \frac{\sum y_i}{n} \\ s_1^2 &= \frac{\sum(x_i - \bar{x})^2}{n} \quad s_2^2 = \frac{\sum(y_i - \bar{y})^2}{n} \\ r &= \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{ns_1 s_2} \end{aligned}$$

All we have to do now is to figure out the volume of the $2n$ dimensional region filled by taking each of the five quantities s_1, s_2, $\bar{x}$, $\bar{y}$ and r, and increasing them by ds_1, ds_2, $d\bar{x}$, $d\bar{y}$ and dr respectively.

We know that

$$\frac{\sum x_i}{\sqrt{n}} = \sqrt{n}\bar{x}$$

is a plane in the 'first' n dimensions (the x-space), going through $(\bar{x}, \bar{x}, ...\bar{x})$ and having $\frac{(1,1,...1)}{\sqrt{n}}$ as normal. Similarly

$$\sum (x_i - \bar{x})^2 = n s_1^2$$

is an n dimenstional sphere centered on $(\bar{x}, \bar{x}, ...\bar{x})$, with the radius of $s_1\sqrt{n}$. The plane cuts the sphere in 'the middle', thus creating (the corresponding cross section) an $n-1$ dimensional sphere, whose surface is

$$\frac{2\pi^{(n-1)/2} s_1^{n-2} n^{(n-2)/2}}{\Gamma\left(\frac{n-1}{2}\right)}$$

This needs to be further multiplied (since the $\sqrt{n}ds_1$ and $\sqrt{n}d\bar{x}$ directions are perpendicular) by $n\ d\bar{x}\ ds_1$.

Similar argument can be made in the y space, except now we need to keep the $(y_1 - \bar{y}, y_2 - \bar{y}, ...y_n - \bar{y})$ vector at a fixed angle to $(x_1 - \bar{x}, x_2 - \bar{x}, ...x_n - \bar{x})$, an angle whose $\cos$ is r. This reduces the $n-1$ dimensional shperical surface of radius $s_2\sqrt{n}$ to an $n-2$ dimensional spherical surface of radius $s_2\sqrt{n(1-r^2)}$. This then contributes

$$\frac{2\pi^{(n-2)/2} s_1^{n-3} n^{(n-3)/2} (1-r^2)^{(n-3)/2}}{\Gamma\left(\frac{n-2}{2}\right)} n\ d\bar{y}\ ds_2\ \frac{s_2\sqrt{n}}{\sqrt{1-r^2}} dr$$

So now we have

$$\frac{n^n (1-r^2)^{(n-4)/2} s_1^{n-2} s_2^{n-2}}{2^{n-2}\pi^{3/2}(1-\rho^2)^{n/2}\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}$$
$$\cdot \exp\left(-\frac{n(s_1^2 + \bar{x}^2 + s_2^2 + \bar{y}^2 - 2\rho r s_1 s_2 - 2\rho\bar{x}\bar{y})}{2(1-\rho^2)}\right) d\bar{x}d\bar{y}ds_1 ds_2 dr$$

(R. A. Fisher, 1915).

The $d\bar{x}d\bar{y}$ integration (separable, from $-\infty$ to ∞ each) can be carried out easily, yielding

$$\frac{n^{n-1}(1-r^2)^{(n-4)/2} s_1^{n-2} s_1^{n-2}}{\pi(1-\rho^2)^{(n-1)/2}(n-3)!} \cdot$$
$$\exp\left(-\frac{n(s_1^2 + s_2^2 - 2\rho r s_1 s_2)}{2(1-\rho^2)}\right) ds_1 ds_2 dr$$

(we have also replaced $2^{n-3}\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)$ by $(n-3)!\sqrt{\pi}$).

To integrate over s_1 and s_2, we realize that the $s_1 > s_2$ and $s_1 < s_2$ regions must contribute the same amount (the function is $s_1 \leftrightarrow s_2$ symmetric), so we multiply the integrand by 2 and integrate over the latter.

Introducing the following one-to-one transformation (ξ goes from 0 to ∞, η from 1 to infinity):

$$\begin{aligned} \xi &= s_1 s_2 \\ \eta &= \frac{1}{2}\left(\frac{s_1}{s_2}+\frac{s_2}{s_1}\right) = \frac{s_1^2+s_2^2}{2s_1s_2} \end{aligned}$$

whose Jacobian is

$$\det\left[\begin{array}{cc} s_2 & s_1 \\ \frac{1}{2}\left(\frac{1}{s_2}-\frac{s_2}{s_1^2}\right) & \frac{1}{2}\left(\frac{1}{s_1}-\frac{s_1}{s_2^2}\right) \end{array}\right] =$$
$$\left(\frac{s_2}{s_1}-\frac{s_1}{s_2}\right) = 2\sqrt{\eta^2-1}$$

results in

$$\frac{n^{n-1}(1-r^2)^{(n-4)/2}\xi^{n-2}}{\pi(1-\rho^2)^{(n-1)/2}(n-3)!}\exp\left(-\frac{n(\eta-\rho r)\xi}{1-\rho^2}\right)\frac{d\xi d\eta}{\sqrt{\eta^2-1}}dr$$

which can be easily integrated over ξ, getting

$$\frac{(n-2)(1-r^2)^{(n-4)/2}}{\pi(1-\rho^2)^{(n-1)/2}}\cdot\left(\frac{1-\rho^2}{\eta-\rho r}\right)^{n-1}\frac{d\eta}{\sqrt{\eta^2-1}}dr$$

And the last substitution is

$$\eta = \frac{1-\rho r z}{1-z}$$

(z going from 0 to 1) replaces $\eta-\rho r$ by

$$\frac{1-\rho r}{1-z}$$

$d\eta$ by

$$\frac{1-\rho r}{(1-z)^2}dz$$

and $\sqrt{\eta^2-1}$ by

$$\frac{\sqrt{2z(1-\rho r)}\cdot\sqrt{1-\frac{1+\rho r}{2}z}}{1-z}$$

getting

$$\frac{(n-2)(1-r^2)^{(n-4)/2}z^{-1/2}\left(1-\frac{1+\rho r}{2}z\right)^{-1/2}}{\sqrt{2}\pi}$$
$$\cdot\frac{(1-\rho^2)^{(n-1)/2}(1-z)^{n-2}}{(1-\rho r)^{n-3/2}}dz\,dr$$

Maple tells us that

$$\int_0^1 z^k(1-z)^m(1-az)^\ell dz =$$
$$\frac{\Gamma(k+1)\Gamma(m+1)}{\Gamma(k+m+2)}F(-\ell,k+1;m+k+2;a)$$

which in our case means

$$\frac{\Gamma(\frac{1}{2})\Gamma(n-1)}{\Gamma(n-\frac{1}{2})}F(\tfrac{1}{2},\tfrac{1}{2};n-\tfrac{1}{2};\tfrac{1+\rho r}{2})$$

The final answer is thus

$$\frac{(n-2)(n-2)!(1-\rho^2)^{(n-1)/2}}{\sqrt{2\pi}\Gamma(n-\frac{1}{2})}\cdot$$

$$\frac{(1-r^2)^{(n/2-2)}}{(1-\rho r)^{n-3/2}}F(\tfrac{1}{2},\tfrac{1}{2};n-\tfrac{1}{2};\tfrac{1+\rho r}{2})dr$$