Let us assume that we have a RIS of size n from an exponential distribution with the mean of β. Since we know that the distribution of $\sum_{i=1}^{n} X_i$ is $\mathsf{gamma}(n, \beta)$, we can easily write down the *exact* PDF of

$$Z \equiv \frac{\bar{X} - \beta}{\frac{\beta}{\sqrt{n}}}$$

namely

$$f(z) = \frac{(n + z\sqrt{n})^{n-1} \exp^{-n - z\sqrt{n}}}{\Gamma(n)} \cdot \sqrt{n}$$

(note that β has cancelled out). It is easier to work with

$$\ln f(z) = (n-1)\ln(n + z\sqrt{n}) - n - z\sqrt{n} + \ln\sqrt{n} - \ln\Gamma(n)$$

We want to expand this function in powers of $w \equiv \frac{1}{\sqrt{n}}$, up to and including the w^2 term.

This can be achieved with the help of the following (Stirling's) formula:

$$\ln\Gamma(n) \simeq n(\ln n - 1) - \ln\sqrt{n} + \ln\sqrt{2\pi} + \frac{1}{12n} - \frac{1}{36n^3} + ...$$

Since $\ln(n + z\sqrt{n}) = \ln(1 + z\ w) - 2\ln w$, we can expand $\ln f(z)$ as follows

$$\left(\frac{1}{w^2} - 1\right)\ln(1+z\ w) - 2\left(\frac{1}{w^2} - 1\right)\ln w - \frac{1}{w^2} - \frac{z}{w} - \ln w - \frac{1}{w^2}(-2\ln w - 1) + \ln w - \ln\sqrt{2\pi} - \frac{w^2}{12} + ...$$

Furthermore, since

$$\ln(1 + z\ w) \simeq z\ w - \frac{z^2 w^2}{2} + \frac{z^3 w^2}{3} - \frac{z^4 w^4}{4} + ...$$

one can easily verify that all singularities of the previous expression cancel out, and we are left with

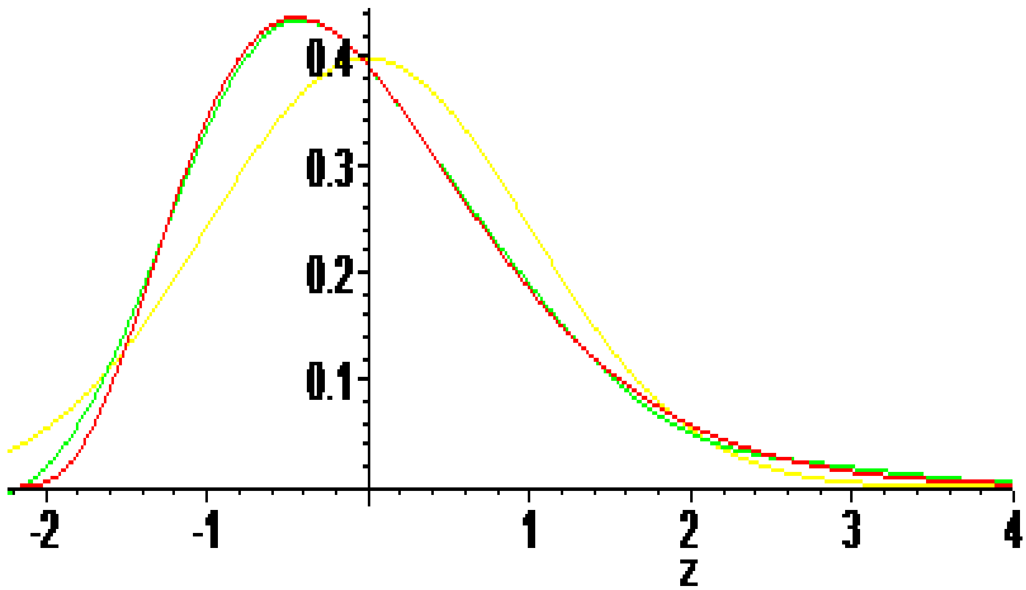
$$-\frac{z^2}{2} - \ln\sqrt{2\pi} + \frac{z^3 - 3z}{3} w - \frac{3z^4 - 6z^2 + 1}{12} w^2 + ...$$

This means that we are getting the following approximation for

$$\begin{aligned} f(z) &\simeq \frac{\exp(-\frac{z^2}{2})}{\sqrt{2\pi}}\left(1 + \frac{z^3 - 3z}{3\sqrt{n}} - \frac{3z^4 - 6z^2 + 1}{12n} + \frac{(z^3 - 3z)^2}{2 \cdot 3^2 n} + ...\right) \\ &= \frac{\exp(-\frac{z^2}{2})}{\sqrt{2\pi}}\left(1 + \frac{z^3 - 3z}{3\sqrt{n}} + \frac{2z^6 - 21z^4 + 36z^2 - 3}{36n} + ...\right) \end{aligned}$$

This is a fairly accurate approximation to the original (exact) PDF, even when

n is as small as 5, as the following graph indicates:



(the exact distribution is in red, the basic Normal approximation is in yellow).

Similarly, we can convert the exact probability function of the Poisson distribution, namely

$$\frac{\lambda^x}{\Gamma(1+x)} \cdot e^{-\lambda}$$

into the PDF of

$$Z \equiv \frac{X-\lambda}{\sqrt{\lambda}}$$

getting

$$f(z) = \frac{\lambda^{\lambda+z\sqrt{\lambda}}}{\Gamma\left(1+\lambda+z\sqrt{\lambda}\right)} \cdot e^{-\lambda} \cdot \sqrt{\lambda}$$

Now, introducing $\lambda = \frac{1}{w^2}$ and taking ln of the previous expression, we get

$$\ln f(z) = -2\left(\frac{1}{w^2}+\frac{z}{w}\right)\ln w - \frac{1}{w^2} - \ln w - \ln\Gamma\left(1+\frac{1}{w^2}+\frac{z}{w}\right)$$

(note that this is still an exact result). Now, realizing that

$$\ln\Gamma(1+m) = \ln m + \ln\Gamma(m) \simeq m(\ln m - 1) + \ln\sqrt{m} + \ln\sqrt{2\pi} + \frac{1}{12m} - \frac{1}{36m^3} + ...$$

we can expand $\ln\Gamma\left(1+\frac{1}{w^2}+\frac{z}{w}\right)$ as follows

$$\simeq \left(\frac{1}{w^2}+\frac{z}{w}\right)[\ln(1+w\ z) - 2\ln w - 1] + \frac{1}{2}[\ln(1+w\ z) - 2\ln w] + \ln\sqrt{2\pi} + \frac{w^2}{12(1+z\ w)} + ...$$

Substituting this into $\ln f(z)$, and further expanding $\ln(1 + w\ z)$, we can again easily verify that all singularities cancel, and that we get

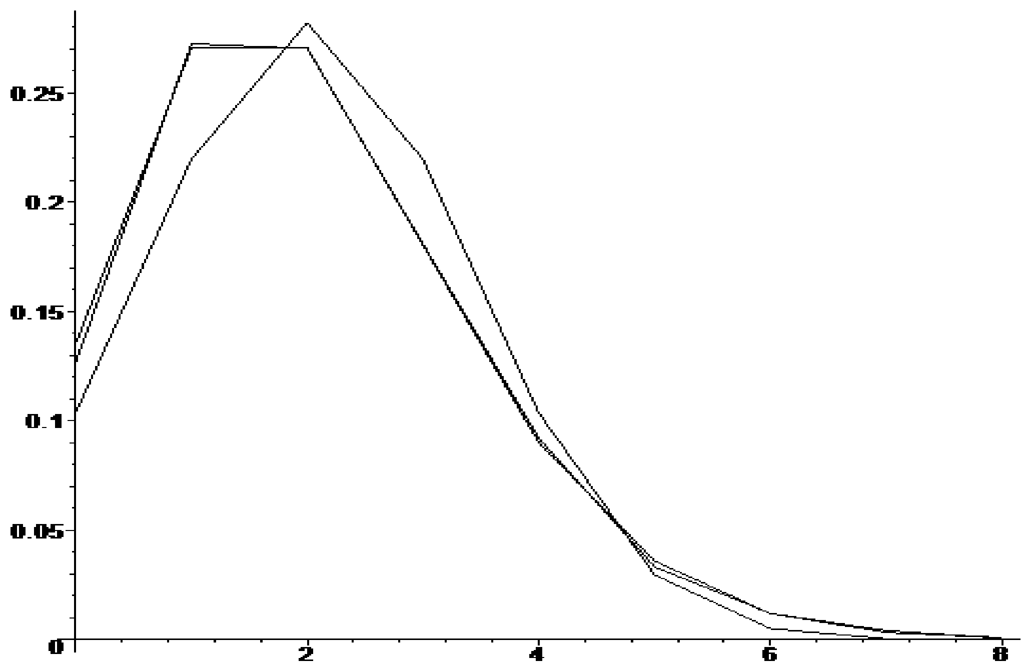
$$\ln f(z) \simeq -\frac{z^2}{2} - \ln\sqrt{2\pi} + \frac{z^3-3z}{6}w - \frac{z^4-3z^2+1}{12}w^2 + ...$$

implying that

$$\begin{aligned} f(z) &\simeq \frac{\exp(-\frac{z^2}{2})}{\sqrt{2\pi}}\left(1+\frac{z^3-3z}{6\sqrt{\lambda}} - \frac{z^4-3z^2+1}{12\lambda} + \frac{(z^3-3z)^2}{2\cdot 6^2\lambda} + ...\right) \\ &= \frac{\exp(-\frac{z^2}{2})}{\sqrt{2\pi}}\left(1+\frac{z^3-3z}{6\sqrt{\lambda}} + \frac{z^6-12z^4+27z^2-6}{72\lambda} + ...\right) \end{aligned}$$

This can be converted back to the x scale by $z \to \frac{x-\lambda}{\sqrt{\lambda}}$, and dividing the resulting expression by $\sqrt{\lambda}$. We can then readily compare the exact Poisson probabilities with those computed based on this (and the basic Normal) approximation.

The results (this time, expressed in term of the corresponding errors) are displayed, for $\lambda = 2$, in the following graph:



One can see that the new approximation is hardly distinguishable from the exact probabilities, while the Normal curve is way off (that's why we would never consider using it as approximation, unless $\lambda \geq 30$).