1 Solving a normal equation for a MLE

Suppose the normal equation reads

$$\overline{G(X,\hat{\mu})} = 0$$

where $\hat{\mu}$ is the MLE of μ.

This can be expanded to

$$\overline{g_0} + \overline{g_1} \cdot d + \overline{g_2} \cdot d^2 + \overline{g_3} \cdot d^3 = 0 \tag{1}$$

where

$$g_i \equiv \frac{G^{(i)}(X,\mu)}{i!}$$
$$d \equiv \mu_1 \varepsilon + \mu_2 \varepsilon^2 + \mu_3 \varepsilon^3$$

$G^{(i)}$ is the i^{th} derivative of G with respect to its second argument, μ is the true value of μ, and $d = \hat{\mu} - \mu$ (expanded to include the $\frac{1}{\sqrt{n}}$, $\frac{1}{n}$ and $\frac{1}{n^{3/2}}$ proportionate corrections - ε is the corresponding 'marker' whose value is 1).

Replacing each g_i by $\varepsilon \cdot (g_i - m_i) + m_i \equiv \varepsilon \cdot U_i + m_i$, where $m_i \equiv \mathbb{E}(g_i)$ - note that m_0 must be identically 0 (it is useful to verify it, as a 'warm up' exercise) - we can expand the LHS of (1) in ε, getting:

$$\left(m_1\mu_1 + \overline{U_0}\right)\cdot\varepsilon + \left(m_1\mu_2 + \overline{U_1}\cdot\mu_1 + m_2\mu_1^2\right)\cdot\varepsilon^2 + \left(\mu_3\cdot m_1 + \overline{U_1}\cdot\mu_2 + \overline{U_2}\cdot\mu_1^2 + 2m_2\cdot\mu_1\mu_2 + m_3\mu_1^3\right)\cdot\varepsilon^3$$

Making each coefficient equal to 0 yields the following solution:

$$\begin{aligned}
\mu_1 &= -\frac{\overline{U_0}}{m_1} \\
\mu_2 &= -\frac{m_2\overline{U_0}^2}{m_1^3} + \frac{\overline{U_0}\cdot\overline{U_1}}{m_1^2} \\
\mu_3 &= -\frac{2m_2^2\overline{U_0}^3}{m_1^5} + \frac{3m_2\overline{U_0}^2\cdot\overline{U_1} + m_3\overline{U_0}^3}{m_1^4} - \frac{\overline{U_0}^2\cdot\overline{U_2} + \overline{U_0}\cdot\overline{U_1}^2}{m_1^3}
\end{aligned}$$