1. __

   a. full length of $2^2 \times 5^3$: We can either squeeze the full length out of the $\boxed{5^4}$ subgroup, by meeting this condition:

   $$= 2, 3 \mod 5 \quad \text{but} \; \neq 7, 18 \mod 25$$

   OR we can get the 2^2 length from the $\boxed{2^4}$ subgroup, and the 5^3 length from $\boxed{5^4}$, by

   $$= 3, 5 \mod 8 \;\; \text{AND} \neq 1, 24, 7, 18 \mod 25$$

   (here, we should also add AND $\neq 0 \bmod 5$).

   b. length 125: We have to get the 5^3 length (exactly) from the $\boxed{5^4}$ subgroup, and length 1 from $\boxed{2^4}$, thus

   $$= 6, 11, 16, 21 \mod 24 \;\; \text{AND} \;\; = 1 \mod 16$$

2. For Metropolis simulation of a sample from

   $$f(z) = \frac{\exp\left(-\frac{z^2}{2}\right)}{\sqrt{2\pi}}$$

   we use our standard program with $F(z) = -z$.

3. First, we find the marginal distribution function of X, make it equal to U_1, and solve for X:

   $$f(x) = \frac{1}{(1+x)^2} \qquad F(x) = 1 - \frac{1}{1+x} \qquad X = \frac{U_1}{1-U_1}$$

   Then, we get the conditional distribution function of $Y|X$, make it equal to U_2, and solve for Y:

   $$f(y|x) = e^{x-y} \qquad F(y|x) = 1 - e^{x-y} \qquad Y = X - \ln(1-U_2)$$

4. First, we get the first four cumulants of $\mathcal{U}(0,1)$: $\kappa_1 = \frac{1}{2}$, $\kappa_2 = \frac{1}{12}$, $\kappa_3 = 0$ (these, we remember from MATH 2P81) and $\kappa_4 = \frac{-1}{120}$. Secondly, we evaluate the first three derivatives of $g(x) = \frac{1}{1+x^2}$ at $x = \frac{1}{2}$, getting: $g' = \frac{-16}{25}$. $g'' = \frac{-32}{125}$ and $g''' = \frac{2304}{625}$. Then, using our formulas, we find the following four cumulants of $\frac{1}{1+U^2}$:

   $$\begin{aligned} K_1 &= \frac{4}{5} - \frac{4}{375n} \\ K_2 &= \frac{64}{1875n} - \frac{2272}{140625n^2} \\ K_3 &= \frac{-512}{234375n^2} \\ K_4 &= \frac{-303104}{87890625n^3} \end{aligned}$$

   which translates into

   $$\begin{aligned} \Gamma_3 &= -\frac{\sqrt{3}}{5\sqrt{n}} \\ \Gamma_4 &= -\frac{74}{25n} \end{aligned}$$

   The approximation follows easily.

5. This time, $g(x) = \exp(\frac{\alpha}{x})$, implying $g' = -4\alpha e^{2\alpha}$ and $g'' = 16\alpha(1+\alpha)e^{2\alpha}$. Solving $\kappa_3 g' + 3\kappa_2^2 g'' = \frac{\alpha}{3} e^{2\alpha}(1+\alpha) = 0$ yields $\alpha = -1$. Recomputing $g' = 4e^{-2}$ and $g'' = 0$ and

$g''' = -32e^{-2}$, and plugging into our formulas yields:

$$\begin{aligned} K_1 &= e^{-2} \\ K_2 &= \frac{4e^{-4}}{3n} - \frac{8e^{-4}}{9n^2} \\ K_3 &= 0 \\ K_4 &= -\frac{928e^{-8}}{135n^3} \end{aligned}$$

implying

$$\begin{aligned} \Gamma_3 &= 0 \\ \Gamma_4 &= -\frac{58}{15n} \end{aligned}$$

Building the actual PDF is easy.

6. The normal equation is

$$\overline{\frac{X - a_0 - \varepsilon a_1}{1 + (X - a_0 - \varepsilon a_1)^2}} = \overline{\frac{X - a_0}{1 + (X - a_0)^2}} + \varepsilon a_1 \overline{\frac{(X - a_0)^2 - 1}{(1 + (X - a_0)^2)^2}} + ... = 0$$

Since

$$\mathbb{E}\left[\frac{(X - a_0)^2 - 1}{(1 + (X - a_0)^2)^2}\right] = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(x - a_0)^2 - 1}{(1 + (x - a_0)^2)^4} dx = -\frac{1}{2}$$

$$a_1 = 2 \cdot \overline{\frac{X - a_0}{1 + (X - a_0)^2}}$$

whose variance is

$$V = \frac{2}{\pi \cdot n}\int_{-\infty}^{\infty} \frac{4(x - a_0)^2}{(1 + (x - a_0)^2)^4} dx = \frac{1}{2n}$$

7.

$$\mathbb{E}\left(\overline{X^2 - 2}^3 \cdot \overline{\sin X - \tfrac{1}{2}}^2\right)$$

$$\begin{aligned} \mathbb{E}\left(\overline{U - \mu_u}^3 \cdot \overline{V - \mu_v}^2\right) &= \frac{\kappa_{32}}{n^4} + 3\frac{\kappa_{20}\kappa_{12}}{n^3} + 6\frac{\kappa_{11}\kappa_{21}}{n^3} + \frac{\kappa_{02}\kappa_{30}}{n^3} \\ &= \frac{4,634,172}{15,625n^4} + \frac{4,377}{25n^3} \end{aligned}$$

where

$$\begin{aligned} \kappa_{20} &= \mu_{20} = \int_0^{\infty} (x^2 - 2)^2 e^{-x} dx = 20 \\ \kappa_{12} &= \mu_{12} = \int_0^{\infty} (x^2 - 2)(\sin x - \frac{1}{2})^2 e^{-x} dx = \frac{197}{250} \\ \kappa_{11} &= \mu_{11} = \int_0^{\infty} (x^2 - 2)(\sin x - \frac{1}{2}) e^{-x} dx = -\frac{1}{2} \\ \kappa_{21} &= \mu_{21} = \int_0^{\infty} (x^2 - 2)^2 (\sin x - \frac{1}{2}) e^{-x} dx = -13 \\ \kappa_{02} &= \mu_{02} = \int_0^{\infty} (\sin x - \frac{1}{2})^2 e^{-x} dx = \frac{3}{20} \\ \kappa_{30} &= \mu_{30} = \int_0^{\infty} (x^2 - 2)^3 e^{-x} dx = 592 \\ \kappa_{32} &= \int_0^{\infty} (x^2 - 2)^3 (\sin x - \frac{1}{2})^2 e^{-x} dx - \mu_{30}\mu_{02} - 6\mu_{11}\mu_{21} - 3\mu_{20}\mu_{12} = \frac{4634172}{15625} \end{aligned}$$

Secondly, the MGF of $\ln(X_1)$ is

$$\int_0^\infty e^{t\cdot\ln(x)}e^{-x}dx = \int_0^\infty x^t e^{-x}dx = \Gamma(1+t)$$

(this is the Gamma function now - Maple knows how to handle it). Answer:

$$\frac{d^4}{dt^4}\ln\Gamma(1+t)\bigg|_{t=0} = \frac{\pi^4}{15}$$