Please give numerical answers in *decimal* form (to at least 4 significant digits).

1. Consider a sequence generated by

$$x_{n+1} = a \cdot x_n \qquad \bmod \quad m$$

where $x_0 = 11111111111$, $m = 20751953125$, and a is a positive integer less than m.

What is the longest possible cycle we can generate (by a proper choice of a), how many such values of a are there (which will generate the longest cycle), and what set of sufficient and necessary conditions must each such a meet?

2. Consider a RIS of size n from a distribution with the following pdf

$$f(x) = \begin{cases} \sin(x) & 0 < x < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find an $\frac{1}{n}$-accurate approximation to the pdf of

$$Z_n = \frac{\sin \bar{X} - \mathbb{E}\left(\sin \bar{X}\right)}{\sqrt{\mathrm{Var}(\sin \bar{X})}}$$

   including (of course) the corresponding expansions of both $\mathbb{E}\left(\sin \bar{X}\right)$ and $\mathrm{Var}(\sin \bar{X})$.

   (b) Apply the result to accurately estimate the probability of $0.8 < \sin \bar{X} < 0.9$, when $n = 10$.

3. Consider the MLE of $\omega > 0$, based on a RIS of size n from a Cauchy distribution with the following pdf

$$f(x) = \frac{1}{\pi} \cdot \frac{\omega}{\omega^2 + x^2} \qquad\qquad -\infty < x < \infty$$

   (a) We know that the sampling distribution of this estimator is approximately Normal (when n is large). To a sufficient (basic-Normal) accuracy, find the corresponding mean and variance.

   (b) Using this approximation, and assuming that $\omega = 10$ and $n = 50$, estimate the probability that the resulting ML estimator will have a value between 11 and 14.

4. Consider a bivariate distribution with the following joint pdf

$$f(x, y) = \begin{cases} 2x + y + 1 & x > 0, \quad y > 0 \quad \text{and} \quad x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Describe how to generate a RIS from this distribution, based on independent random variables of the $\mathcal{U}(0, 1)$ type (i.e. how do we *transform* U_1 and U_2 into X and Y).

5. Find the basic Normal approximation for the distribution of

$$T \equiv \frac{1}{1 + \bar{X} \cdot \bar{Y}}$$

assuming that (X_i, Y_i), where i goes from 1 to n, is a RIS from the bivariate distribution of the *previous question* (and $\bar{X}$ and $\bar{Y}$ are the corresponding sample means).

Based on this approximation, estimate $\Pr(0.87 < T < 0.91)$ when $n = 35$.

6. This is a continuation of the previus question. If we now define

$$U = \exp(\bar{X} - \bar{Y})$$

find the basic (bivariate) Normal approximation to the distribution of T and U.

Also: Transform T and U into standardized and *independent* (at this level of accuracy) Z_1 and Z_2.