MATH 5P88 Final Exam, 2011. Full credit given for 5 complete and correct answers.

1. Assuming that $x_0 = 6,427$ and

$$x_{n+1} = a \cdot x_n \mod 10,000$$

spell out the exact (necessary and sufficient) conditions for a to generate

a. the longest possible sequence,

b. a sequence of length 125.

2. Using Monte-Carlo simulation with Metropolis, generate a single block of 100 iterations with 100 configurations from the (univariate) distribution whose PDF is

$$\frac{\exp\left(-\frac{z^2}{2}\right)}{\sqrt{2\pi}}$$

(choose your time step $\tau = 0.1$, and discard the first 50 iterations to equilibrate - you will thus need 150 iterations in total). Compute

a. the grand-mean estimate of $\mathbb{E}\left(Z^2\right)$,

b. the first serial correlation of the 100 iteration estimates of $\mathbb{E}\left(Z^2\right)$,

c. the average (Metropolis) rejection rate (exclude equilibration),

d. the maximum staleness encountered throughout the simulation (exclude equilibration).

3. Suppose U_1 and U_2 are independent random variables, uniformly distributed over the $(0, 1)$ interval. Design a transformation of U_1 and U_2 to yield another pair of random variables, say X and Y, with the following joint PDF:

$$\frac{e^{x-y}}{(1+x)^2} \quad 0 < x < y$$

4. Suppose U_1, U_2, ...U_n is a random independent sample from the uniform distribution in the $(0, 1)$ interval. Find the $\frac{1}{n}$ accurate approximation to the PDF of

$$\frac{1}{1+\overline{U}^2}$$

(*not* of $\bar{U}$), where $\overline{U}$ is the corresponding sample mean (highlight the values of the first four cumulants of this function of $\bar{U}$). Plot the error if this approximation using $n = 10$ (note that the range of the new random variable is $\frac{1}{2}$ to 1).

5. This is a continuation of the previous question. Find α such that the distribution of

$$\exp\left(\frac{\alpha}{\overline{U}}\right)$$

is 'nearly' (i.e., up to the same $\frac{1}{n}$ approximation) symmetric, and construct the corresponding approximate PDF (again, of the resulting *function* of $\bar{U}$, *not* of $\bar{U}$ itself; highlight the cumulants). Plot the error of this approximation when $n = 10$.

6. Consider a random independent sample of size n from a distribution with the following PDF

$$\frac{2}{\pi\left(1+(x-a)^2\right)^2}$$

(x can have any real value). Find the basic Normal approximation for the distribution of the MLE of a. Using the following random independent sample from this distribution, compute the corresponding estimate of a, and its standard error: $0.18, 0.00, 0.04, 0.26, 0.39, 0.73, -1.36, 0.68, 0.47, 0.73$.

7. Consider a random independent sample of size n from an exponential distribution with a mean of 1. Find (without any approximation)

$$\mathbb{E}\left(\overline{X^2-2}^3 \cdot \overline{\sin X - \tfrac{1}{2}}^2\right)$$

Hint: First, find a general formula for

$$\mathbb{E}\left(\overline{U-\mu_u}^3 \cdot \overline{V-\mu_v}^2\right)$$

Also, find κ_4 of $\ln X_1$ (X_1 is one single value from the same exponential distribution).