1. Show how to transform U_1 and U_2 (independent, uniform between 0 and 1) into X and Y with the following joint PDF:

$$f(x,y) = \frac{1}{\left(1+\ln\dfrac{y}{x}\right)^2 \cdot (1-\ln y)^2 \cdot x \cdot y} \quad \text{when} \quad x < y < 1 \text{ and } \quad 0 < x < 1 \quad (0 \text{ otherwise})$$

2. Assuming a RIS of size n from the bivariate distribution of the previous question, find the following joint central moment

$$\mathbb{E}\left((\bar{X}-\mu_x)^3 \cdot (\bar{Y}-\mu_y)^3\right)$$

expanded in powers of $\frac{1}{n}$ (please give the coefficients of the expansion in decimal - that goes for the next two questions as well). Also spell out the (decimal) values of $\mu_x \equiv \mathbb{E}(X)$ and $\mu_y \equiv \mathbb{E}(Y)$. Hint: First compute the following decimal expansion:

$$M_{x,y}(t,u) = \sum_{i=0}^{3}\sum_{j=0}^{3} \frac{\mathbb{E}\left(X^i Y^j\right)}{i!j!} t^i u^j$$

3. Assuming a RIS of size n from the bivariate distribution of Question 1, find the $\frac{1}{n}$-accurate approximation to the PDF of

$$U \equiv \left(1 + \frac{n}{\sum_{i=1}^{n} \dfrac{X_i}{1+Y_i}}\right)^{-1}$$

Use this approximation and $n = 15$ to evaluate $\Pr(0.09 < U < 0.12)$. Hint: First express U in terms of the bar (sample-mean) notation. Then, compute the following decimal expansion:

$$M(t) = \sum_{i=0}^{3} \frac{\mathbb{E}\left[\left(\frac{X}{1+Y}\right)^i\right]}{i!} t^i$$

4. Assuming a RIS of size n from the bivariate distribution of Question 1, find the Normal approximation to the *joint* PDF of

$$V \equiv \frac{\bar{X}-\bar{Y}}{\bar{X}+\bar{Y}}$$

and

$$W \equiv \exp\left(\bar{X}+\bar{Y}\right)$$

Use this approximation and $n = 30$ to evaluate $\Pr(V < -0.4 \cap W > 1.7)$.

5. What set of conditions must a meet to generate the longest possible sequence using

$$x_n = a \cdot x_{n-1} \quad \mod 1257565061957837936381$$

where $x_0 = 1$? What is the length of such a sequence? How many values of a are there which generate a sequence of this length? Find one such a (make it 11 digits long).