All numerical results should be given in decimal form (correct to at least 4 significant digits).

1. Consider the following tri-variate PDF (of random variables X, Y and Z):

$$\frac{x+y+z}{3} \cdot e^{-x-y-z} \qquad \text{when} \qquad x>0,\ y>0 \text{ and } z>0,$$

zero otherwise.

   (a) Define a new random variable $W = X + Y + Z$ and compute $\Pr(W \le w)$ for any real w. Convert the answer to the corresponding PDF of W and identify the resulting distribution.

   (b) How would you generate a random value of W based on *four* independent U_1, U_2, U_3 and U_4, each from $\mathcal{U}(0,1)$.

   (c) Find the (marginal) PDF of X and identify the corresponding distribution (hint: it will be a *mixture* of two distributions - spell out the two weights as well).

   (d) Generate a random independent sample of 300 values of X (using results of Part c) and plot the corresponding empirical PDF of X, namely

$$\frac{1}{400}\sum_{i=1}^{300} \max\left(0, 1-(x-X_i)^2\right) + \frac{1}{400}\sum_{i=1}^{300} \max\left(0, 1-(x+X_i)^2\right)$$

   for x from 0 to 6, together (in the same graph) with the exact PDF of X.

2. Generate an initial ensemble of 100 configurations generated, randomly and independently from the uniform distribution over the $1 < x < 2$, $1 < x < 2$ and $1 < z < 2$ cube. Then, advance this ensemble over 200 iteration of Metropolis sampling (use the time step of 0.03) in a way which yields (after equilibration) samples from the trivariate distribution of the previous question. Plot the iteration estimates of

$$\mathbb{E}\left(\frac{1}{1+X\cdot Y\cdot Z}\right)$$

   and (separately) the iteration percentages of rejected moves.

3. Consider the multiplicative group mod 8550986578849. What is the order of 2924207? What set of conditions must an element of this group meet to be of exactly this order? How many group elements of this order are there? Find one such element (hint: is it feasible to look for it randomly?).

4. Let X have a distribution with the following PDF:

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{x^2 \exp\left(-\frac{x^2}{2a^2}\right)}{a^3}$$

when $x > 0$ (zero otherwise) and a is a positive parameter. Find

(a) the first four cumulants of this distribution,

(b) a function g which makes the ($\frac{1}{\sqrt{n}}$ term of) skewness of $g(\bar{X})$ equal to zero ($\bar{X}$ is a sample mean of n observation from the above distribution),

(c) a $\frac{1}{n}$-accurate PDF of $Y = g(\bar{X})$, where g is the function just found.

(d) Using this approximation, compute $\Pr(Y > 2.5)$ when $n = 15$ and $a = 3.2$.

5. Consider a random independent sample of size n from the trivariate distribution of Question 1.

(a) Find a formula for

$$\mathbb{E}\left((\overline{X} - \mu_X)^3 \cdot (\overline{\ln Y} - \mu_{\ln Y})^2\right)$$

where

$$\begin{aligned} \overline{X} &= \frac{\sum_{i=1}^n X_i}{n} \\ \overline{\ln Y} &= \frac{\sum_{i=1}^n \ln(Y_i)}{n} \end{aligned}$$

(quote the two means as well).

(b) Use the basic Normal approximation to estimate

$$\Pr\left(\frac{\overline{Y^2}}{1 + \overline{X}^2} < 1.37\right)$$

when $n = 100$.