Digital answers must be quoted to at least 4 significant digits. Final answers must be entered in your booklet in 'human' form, no Maplese!

1. The tri-variate distribution of X, Y and Z is uniform over the unit ball (a solid sphere) of radius 1, centered on the origin. Find the joint PDF of R, Θ and Ψ (the usual spherical coordinates) defined by

$$\begin{aligned} X &= R\sin\Theta\cos\Psi \\ Y &= R\sin\Theta\sin\Psi \\ Z &= R\cos\Theta \end{aligned}$$

   With the help of this result, express X, Y and Z as a transformation of three independent RVs U_1, U_2 and U_3, each uniformly distributed over the $(0,1)$ interval (simplify your answer).

2. Using Metropolis sampling (and a step size of 0.3) generate a sequence of 200 (after equilibration) random independent (within, but not between) samples of size 100 from the bivariate Normal distribution (call the two RVs X and Y) with both means equal to 0, both variances equal to 1, and the covariance equal to -0.83. Plot the individual sample estimates of $\mathbb{E}\left(\sqrt{X^2+Y^2}\right)$. Compare the grand mean (after equilibration) of your iteration estimates with the theoretical value. Using `with(Statistics)` in this question is permitted.

3. Consider the multiplicative group mod $1,000,000,000,000$. What is the order of $a = 512,001$? How many group elements of this order are there? What set of conditions must an element of this group meet to be of exactly this order? Repeat with $a = 512,061$.

4. Find the $\frac{1}{n}$ accurate approximation for the PDF of $\bar{X}$ when sampling from a distribution with the following PDF

$$f(x) = \frac{\exp\left(-\dfrac{(x-a)^2}{2a^2x}\right)}{\sqrt{2\pi x^3}} \qquad \text{when } x > 0 \qquad \text{(zero otherwise)}$$

   where a is a positive parameter. Leave the answer in the z scale, after you have defined your Z. Repeat with $g(\bar{X})$, where g is a parameter-free function which removes the skewness of $g(\bar{X})$.

5. Find the joint MGF of two RVs X and Y having the following bivariate PDF

$$f(x,y) = 2\exp(-x-y) \qquad \text{when} \qquad 0 < x < y$$

   (zero otherwise). Assuming a RIS of size n from this distribution, derive a formula for computing

$$\mathbb{E}\left((\overline{X} - \mu_X)^2 \cdot (\overline{\ln Y} - \mu_{\ln Y})^2 \cdot (\overline{X^2} - \mu_{X^2})\right)$$

   Also, find the basic Normal approximation for the distribution of

$$\sqrt{\overline{X^2}} \cdot \exp\left(\overline{\ln Y}\right)$$