Constructing cumulants of $g(\bar{X})$

First, quite trivially

$$g(\bar{X}) = g(\mu) + g'(\mu) \cdot \overline{X - \mu} + \frac{g''(\mu)}{2} \cdot \overline{X - \mu}^2 + \frac{g'''(\mu)}{3!} \cdot \overline{X - \mu}^3 + \ldots \qquad (1)$$

Secondly, we need the following table

$$\begin{aligned}
\mathbb{E}\left(\overline{X - \mu}\right) &= 0 \\
\mathbb{E}\left(\overline{X - \mu}^2\right) &= \frac{\sum_{i=1}^n \mathbb{E}\left[(X_i - \mu)^2\right]}{n^2} = \frac{\sigma^2}{n} \\
\mathbb{E}\left(\overline{X - \mu}^3\right) &= \frac{\sum_{i=1}^n \mathbb{E}\left[(X_i - \mu)^3\right]}{n^3} = \frac{\mu_3}{n^2} = \frac{\kappa_3}{n^2} \\
\mathbb{E}\left(\overline{X - \mu}^4\right) &= \frac{\sum_{i=1}^n \mathbb{E}\left[(X_i - \mu)^4\right] + \binom{4}{2,2} \sum_{i<j}^n \mathbb{E}\left[(X_i - \mu)^2 (X_j - \mu)^2\right]}{n^4} \\
&= \frac{n\mu_4 + 3n(n-1)\sigma^4}{n^4} = \frac{3\sigma^4}{n^2} + \frac{\kappa_4}{n^3}
\end{aligned}$$

At this point, just to better understand the procedure, we will similarly expand

$$\begin{aligned}
\mathbb{E}\left(\overline{X}^4\right) &= \frac{\begin{array}{c}\sum_{i=1}^n \mathbb{E}(X_i{}^4) + \binom{4}{3,1} \sum_{i \neq j}^n \mathbb{E}(X_i{}^3 X_j) + \binom{4}{2,1,1} \sum_{i \neq j < k}^n \mathbb{E}(X_i{}^2 X_j X_k) \\ + \binom{4}{2,2} \sum_{i<j}^n \mathbb{E}(X_i{}^2 X_j^2) + \binom{4}{1,1,1,1} \sum_{i<j<k<\ell}^n \mathbb{E}(X_i X_j X_k X_\ell)\end{array}}{n^4} \\
&= \frac{\begin{array}{c} n\dot{\mu}_4 + 4n(n-1)\dot{\mu}_3\mu + 6n(n-1)(n-2)\dot{\mu}_2\mu^2 \\ + 3n(n-1)\dot{\mu}_2^2 + n(n-1)(n-2)(n-3)\mu^4 \end{array}}{n^4} \\
&= \mu^4 + 6\mu^2 \frac{\dot{\mu}_2 - \mu^2}{n} + \frac{4\mu\dot{\mu}_3 + 3\dot{\mu}_2^2 - 18\mu^2\dot{\mu}_2 + 11\mu^4}{n^2} \\
&\quad + \frac{\dot{\mu}_4 - 4\mu\dot{\mu}_3 - 3\dot{\mu}_2^2 + 12\mu^2\dot{\mu}_2 - 6\mu^4}{n^3}
\end{aligned}$$

where $\dot{\mu}_i$ are now the *simple* moments. This time, a good way to verify the correctness of the expansion is to do

$$n + 4n(n-1) + 6n(n-1)(n-2) + 3n(n-1) + n(n-1)(n-2)(n-3) = n^4$$

Now, to continue with our original table

$$\begin{aligned}
\mathbb{E}\left(\overline{X-\mu}^5\right) &= \frac{\sum_{i=1}^n \mathbb{E}\left[(X_i-\mu)^5\right] + \binom{5}{3,2}\sum_{i\neq j}^n \mathbb{E}\left[(X_i-\mu)^3(X_j-\mu)^2\right]}{n^5} \\
&= \frac{n\mu_5 + 10n(n-1)\mu_3\sigma^2}{n^5} = 10\frac{\kappa_3\sigma^2}{n^3} + \frac{\kappa_5}{n^4} \\
\mathbb{E}\left(\overline{X-\mu}^6\right) &= \frac{\begin{array}{c}\sum_{i=1}^n \mathbb{E}\left[(X_i-\mu)^6\right] + \binom{6}{3,3}\sum_{i<j}^n \mathbb{E}\left[(X_i-\mu)^3(X_j-\mu)^3\right] \\ \binom{6}{4,2}\sum_{i\neq j}^n \mathbb{E}\left[(X_i-\mu)^4(X_j-\mu)^2\right] + \binom{6}{2,2,2}\sum_{i<j<k}^n \mathbb{E}\left[(X_i-\mu)^2(X_j-\mu)^2(X_k-\mu)^2\right]\end{array}}{n^6} \\
&= \frac{n\mu_6 + 10n(n-1)\mu_3^2 + 15n(n-1)\mu_4\sigma^2 + 15n(n-1)(n-2)\sigma^6}{n^6} \\
&= 15\frac{\sigma^6}{n^3} + \frac{15\sigma^2\kappa_4 + 10\kappa_3^2}{n^4} + \frac{\kappa_6}{n^5}
\end{aligned}$$

Note that this is the *general* relationship between central moment and cumulants (see below). Also note that the coefficients represent the corresponding number of partitions, e.g.

$$15 = \frac{\binom{6}{2,2,2}}{3!} \qquad 15 = \binom{6}{2,4} \qquad 10 = \frac{\binom{6}{3,3}}{2!} \qquad 1 = \binom{6}{6}$$

Now, taking the expected value of (1), to the $\frac{1}{n}$ accuracy, yields

$$\mu_{(g)} = g(\mu) + \frac{g''(\mu)\cdot\sigma^2}{2n} + \ldots$$

Similarly, the variance of $g(\bar{X})$ is, to the $\frac{1}{n^2}$ accuracy

$$\begin{aligned}
&\mathbb{E}\left[\left(g'(\mu)\cdot\overline{X-\mu} + \frac{g''(\mu)}{2}\cdot\overline{X-\mu}^2 + \frac{g'''(\mu)}{3!}\cdot\overline{X-\mu}^3 - \frac{g''(\mu)\cdot\sigma^2}{2n}\right)^2\right] \\
= \; & \frac{(g')^2\sigma^2}{n} + \frac{g'g''\kappa_3}{n^2} + \frac{g'g''' + \frac{3}{4}(g'')^2}{n^2}\sigma^4 - \frac{(g'')^2\sigma^4}{2n^2} + \frac{(g'')^2\sigma^4}{4n^2} \\
= \; & \frac{(g')^2\sigma^2}{n} + \frac{g'g''\kappa_3 + g'g'''\sigma^4 + \frac{1}{2}(g'')^2\sigma^4}{n^2} + \ldots
\end{aligned}$$

Now, the third central moment (to the $\frac{1}{n^2}$ accuracy)

$$\begin{aligned}
&\mathbb{E}\left[\left(g'(\mu)\cdot\overline{X-\mu} + \frac{g''(\mu)}{2}\cdot\overline{X-\mu}^2 - \frac{g''(\mu)\cdot\sigma^2}{2n}\right)^3\right] \\
= \; & \frac{(g')^3\kappa_3}{n^2} + 3\frac{(g')^2 g''}{2}\cdot\frac{3\sigma^4}{n^2} - 3\frac{(g')^2 g''\sigma^4}{2n^2} \\
= \; & \frac{(g')^2(g'\kappa_3 + 3g''\sigma^4)}{n^2} + \ldots
\end{aligned}$$

and the fourth (to the $\frac{1}{n^3}$ accuracy)

$$\begin{aligned}
&\mathbb{E}\left[\left(g'(\mu)\cdot\overline{X-\mu}+\frac{g''(\mu)}{2}\cdot\overline{X-\mu}^2+\frac{g'''(\mu)}{3!}\cdot\overline{X-\mu}^3-\frac{g''(\mu)\cdot\sigma^2}{2n}\right)^4\right]\\
=\;&\frac{3(g')^4\sigma^4}{n^2}+\frac{(g')^4\kappa_4}{n^3}+4\frac{(g')^3g''}{2}\cdot\frac{10\kappa_3\sigma^2}{n^3}\\
&+\left(4\frac{(g')^3g'''}{3!}+6\frac{(g')^2(g'')^2}{4}\right)\cdot\frac{15\sigma^6}{n^3}-4\frac{(g')^3\kappa_3 g''\sigma^2}{2n^3}\\
&+6\frac{(g')^2(g'')^2\sigma^6}{4n^3}-12\frac{(g')^2g''g''\sigma^2}{4n}\cdot\frac{3\sigma^4}{n^2}\\
=\;&\frac{3(g')^4\sigma^4}{n^2}+\frac{(g')^4\kappa_4}{n^3}+18\frac{(g')^3g''\kappa_3\sigma^2}{n^3}+10\frac{(g')^3g'''\sigma^6}{n^3}\\
&+15\frac{(g')^2(g'')^2\sigma^6}{n^3}+...
\end{aligned}$$

The last expression must be still converted to the corresponding cumulant, by subtracting

$$3\left(\frac{(g')^2\sigma^2}{n}+\frac{g'g''\kappa_3+g'g'''\sigma^4+\frac{1}{2}(g'')^2\sigma^4}{n^2}\right)^2$$

getting

$$\frac{(g')^4\kappa_4}{n^3}+12\frac{(g')^3g''\kappa_3\sigma^2}{n^3}+4\frac{(g')^3g'''\sigma^6}{n^3}+12\frac{(g')^2(g'')^2\sigma^6}{n^3}+...$$

More on cumulants

There is a simple relationship between both simple and central moments and cumulants, which should be clear from the following example:

$$\mu_8=\kappa_8+\binom{8}{6,2}\kappa_6\kappa_2+\binom{8}{5,3}\kappa_5\kappa_3+\binom{8}{4,4}\frac{\kappa_4^2}{2!}+\binom{8}{4,2,2}\frac{\kappa_4\kappa_2^2}{2!}+\binom{8}{3,3,2}\frac{\kappa_3^2\kappa_2}{2!}+\binom{8}{2,2,2,2}\frac{\kappa_2^4}{4!} \tag{2}$$

`Proof:` This follows from expading

$$\sum_{i=1}^{8}\frac{\left(\sum_{j=2}^{8}\frac{\kappa_j}{j!}t^j\right)^i}{i!}$$

finding the overall coefficient of t^8, and multiplying it by 8! This results in adding all terms of the type

$$\frac{8!}{i!}\binom{i}{j_1,j_2,...j_\ell}\left(\frac{\kappa_{i_1}}{i_1!}\right)^{j_1}\left(\frac{\kappa_{i_2}}{i_2!}\right)^{j_2}...\left(\frac{\kappa_{i_\ell}}{i_\ell!}\right)^{j_\ell}=\binom{8}{i_1,i_2,...i_\ell}\left(\frac{\kappa_{i_1}}{j_1!}\right)^{j_1}\left(\frac{\kappa_{i_2}}{j_2!}\right)^{j_2}...\left(\frac{\kappa_{i_\ell}}{j_\ell!}\right)^{j_\ell}$$

with $i_1+i_2+...+i_\ell=8$. □

Since (2) is quite general (it applies to *any* random variable, including $\bar{X}$), we can *directly and immediately*:re-write this as:

$$\mathbb{E}\left(\overline{X-\mu}^{8}\right)=\frac{\kappa_8}{n^7}+\frac{\binom{8}{6,2}\kappa_6\kappa_2+\binom{8}{5,3}\kappa_5\kappa_3+\binom{8}{4,4}\frac{\kappa_4^2}{2!}}{n^6}+\frac{\binom{8}{4,2,2}\frac{\kappa_4\kappa_2^2}{2!}+\binom{8}{3,3,2}\frac{\kappa_3^2\kappa_2}{2!}}{n^5}+\frac{\binom{8}{2,2,2,2}\frac{\kappa_2^4}{4!}}{n^4}$$

as the ℓ^{th} cumulant of $\bar{X}$ is $\frac{\kappa_\ell}{n^{\ell-1}}$.

To extend this to the case of two variables, getting the formula for, say, $\mathbb{E}\left(\overline{X-\mu}^{4}\cdot\overline{Y-\mu}^{4}\right)$, we must 'split' each term of (2) accordingly, e.g.

$$\begin{aligned}
28\kappa_6\kappa_2 &= 6\kappa_{42}\kappa_{02}+16\kappa_{33}\kappa_{11}+6\kappa_{24}\kappa_{20}\\
210\kappa_4\kappa_2^2 &= 3\kappa_{40}\kappa_{02}^2+48\kappa_{31}\kappa_{11}\kappa_{02}+72\kappa_{22}\kappa_{11}^2+36\kappa_{22}\kappa_{20}\kappa_{02}+48\kappa_{13}\kappa_{11}\kappa_{20}+3\kappa_{04}\kappa_{20}^2\\
105\kappa_2^4 &= 9\kappa_{20}^2\kappa_{02}^2+72\kappa_{11}^2\kappa_{20}\kappa_{02}+24\kappa_{11}^4
\end{aligned}$$