1. A good choice of τ is in the 0.25 to 0.35 range, the grand mean should be in the 0.377 to 0.384 range, with standard error (which is estimated based on only 5 values, i.e. not very precisely) in the 0.004 to 0.008 range.

2. In the $\boxed{5^{15}}$ subgroup, the number must $= 1 \bmod 5^5$ but $\neq 1 \bmod 5^6$ (i.e. of the $1 + 5^5 k$ type, where k is not divisible by 5).

   In the $\boxed{7^{15}}$ subgroup, the number must $= 1 \bmod 7^5$ but $\neq 1 \bmod 7^6$ (i.e. of the $1 + 7^5 \ell$ type, where ℓ is not divisible by 7).

   Overall, this means that, after subtracting 1, the number (let us call it a) must have exactly five powers of 5 and exactly five powers of 7 (i.e. it has the form of $1 + 35^5 j$, where j is any integer not divisible by 5 and not divisible by 7 - the easiest choice is $1 + 35^5$).

   To verify that we have made a correct (specific) choice of a, compute $a^{35^{10}}$ mod 35^{15} (in a do loop which does $a := a^{35} \bmod 35^{15}$ ten times). The answer must equal to 1. But that is not all: similarly, we have to check that NEITHER $a^{5\times 35^9} \bmod 35^{15}$ NOR $a^{7\times 35^9} \bmod 35^{15}$ are equal to 1.

3. First, we find (either by direct integration, or through MGF) the first 4 cumulants of the distribution:

$$
\begin{aligned}
\mu &= \frac{\alpha}{\alpha+\beta} \\
\sigma^2 &= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
\kappa_3 &= \frac{2\alpha\beta(\alpha-\beta)}{(\alpha+\beta)^3(\alpha+\beta+1)(\alpha+\beta+2)} \\
\kappa_4 &= \frac{6\alpha\beta(\alpha^3 - 2\alpha^2\beta - 2\alpha\beta^2 + \beta^3 + \alpha^2 - 4\alpha\beta + \beta^2)}{(\alpha+\beta)^4(\alpha+\beta+1)(\alpha+\beta+2)(\alpha+\beta+3)}
\end{aligned}
$$

   $\frac{1}{n}$-accurate approximation for $f(\bar{x})$ is the usual

$$
\frac{\exp(-\frac{z^2}{2})}{\sqrt{2\pi}} \left(1 + \frac{\kappa_3(z^3-3z)}{6\sigma^3\sqrt{n}} + \frac{\kappa_4(z^4-6z^2+3)}{24\sigma^4 n} + \frac{\kappa_3^2(z^6-15z^4+45z^2-15)}{72\sigma^6 n}\right) \frac{\sqrt{n}}{\sigma}
$$

   where

$$
z = \frac{\bar{x}-\mu}{\sigma}\sqrt{n}
$$

   Generating 1000 RISs of size 15 and computing their sample means is easy. For the corresponding emirical distribution, using $h = 0.0225$ seems the optimal choice.

4. The normal equation is

$$
\overline{\frac{X-\hat{a}}{1+(X-\hat{a})^2}} = 0
$$

Expanding $\hat{a}$ the usual way (up to and including ε^2), then correspondingly expanding the LHS of the previous equation (in the usual two steps) yields

$$\left(\bar{U} - \frac{\alpha_1}{4}\right)\varepsilon + \left(\alpha_1 \bar{V} - \frac{\alpha_2}{4}\right)\varepsilon^2 + ... = 0$$

where

$$\begin{aligned} \bar{U} &= \overline{\frac{X-\alpha}{1+(X-\alpha)^2}} \\ \bar{V} &= \overline{\frac{2(X-\alpha)^2}{[1+(X-\alpha)^2]^2} - \frac{1}{1+(X-\alpha)^2} + \frac{1}{4}} \end{aligned}$$

For $\hat{a}$, we thus get

$$\alpha + 4\bar{U} + 16\bar{U}\bar{V}$$

implying

$$\begin{aligned} \mu_{\hat{a}} &= \mathbb{E}\left(\alpha + 4\bar{U} + 16\bar{U}\bar{V}\right) = \alpha \\ \sigma^2_{\hat{a}} &= \mathbb{E}\left[(4\bar{U})^2\right] = \frac{2}{n} \\ \kappa_3 &= \mathbb{E}\left[(4\bar{U})^3 + 3(4\bar{U})^2 16\bar{U}\bar{V}\right] = 0 \end{aligned}$$

(on closer inspection one can see that the distribution of $\hat{a}$ must be symmetrical about α - no wonder the exact mean is α, and skewness is zero!). $\frac{1}{\sqrt{n}}$-accurate approximation is, in this case (as for all symmetric distributions) the same as the basic Normal approximation, namely

$$f_{\hat{a}}(x) = \frac{\exp\left(-\frac{n(x-\alpha)^2}{4}\right)}{\sqrt{\frac{4\pi}{n}}}$$

Solving the normal eqaution with the given values of X is easy (with the help of 'fsolve'); since we get 3 roots (but we know that the solution is unique) we have to find which of those 3 roots corresponds to the highest value of the Likelihood function (or its logarithm) - that proves to be (not surprisingly) 1.679. By the way, 87.29 was not a typo - Cauchy distribution is well known for yielding (quite legitimately) 'crazy' values once in a while.

5. The normal equations are

$$\begin{aligned} \Psi(\hat{\alpha}) + \ln \hat{\beta} &= \overline{\ln X} \\ \hat{\alpha}\hat{\beta} &= \overline{X} \end{aligned}$$

Doing the usual ε expansion (this time, only to the ε accuracy) yields

$$\begin{aligned} \Psi(\alpha) + \ln \beta + \left(\alpha_1 \Psi(1,\alpha) + \frac{\beta_1}{\beta}\right)\varepsilon + ... &= \varepsilon \bar{U} + \Psi(\alpha) + \ln \beta \\ \alpha\beta + (\alpha_1 \beta + \alpha \beta_1)\varepsilon + ... &= \varepsilon \bar{V} + \alpha\beta \end{aligned}$$

where

$$\begin{aligned}\bar{U} &= \overline{\ln X - \Psi(\alpha) - \ln \beta} \\ \bar{V} &= \overline{X - \alpha\beta}\end{aligned}$$

Solving:

$$\begin{aligned}\alpha_1 &= \frac{\alpha\beta\bar{U} - \bar{V}}{\beta[\alpha\Psi(1,\alpha) - 1]} \\ \beta_1 &= \frac{-\beta\bar{U} + \Psi(1,\alpha)\bar{V}}{\alpha\Psi(1,\alpha) - 1}\end{aligned}$$

Thus

$$\begin{aligned}\sigma^2_{\hat{\alpha}} &= \frac{\mathbb{E}\left[\left(\alpha\beta\bar{U} - \bar{V}\right)^2\right]}{\beta^2[\alpha\Psi(1,\alpha) - 1]^2} = \frac{\alpha}{n \cdot [\alpha\Psi(1,\alpha) - 1]} \\ \sigma^2_{\hat{\beta}} &= \frac{\mathbb{E}\left[\left(-\beta\bar{U} + \Psi(1,\alpha)\bar{V}\right)^2\right]}{[\alpha\Psi(1,\alpha) - 1]^2} = \frac{\beta^2\Psi(1,\alpha)}{n \cdot [\alpha\Psi(1,\alpha) - 1]} \\ Cov(\hat{\alpha},\hat{\beta}) &= \frac{\mathbb{E}\left[\left(\alpha\beta\bar{U} - \bar{V}\right)\left(-\beta\bar{U} + \Psi(1,\alpha)\bar{V}\right)\right]}{\beta[\alpha\Psi(1,\alpha) - 1]^2} = \frac{-\beta}{n \cdot [\alpha\Psi(1,\alpha) - 1]}\end{aligned}$$

since

$$\begin{aligned}\mathbb{E}\left[\bar{U}^2\right] &= \frac{\Psi(1,\alpha)}{n} \\ \mathbb{E}\left[\bar{V}^2\right] &= \frac{\alpha\beta^2}{n} \\ \mathbb{E}\left[\bar{U}\bar{V}\right] &= \frac{\beta}{n}\end{aligned}$$

This implies that

$$\rho = \frac{-1}{\sqrt{\alpha\Psi(1,\alpha)}}$$

The basic Normal approximation is

$$f_{\hat{\alpha},\hat{\beta}}(x,y) = \frac{\exp\left(-\frac{z_1^2 + z_2^2 - 2\rho z_1 z_2}{2(1-\rho^2)}\right)}{2\pi\sigma_{\hat{\alpha}}\sigma_{\hat{\beta}}\sqrt{1-\rho^2}}$$

where

$$\begin{aligned}z_1 &= \frac{\hat{\alpha} - \alpha}{\sigma_{\hat{\alpha}}} \\ z_2 &= \frac{\hat{\beta} - \beta}{\sigma_{\hat{\beta}}}\end{aligned}$$

Solving the two normal equation with our set of X values is again quite easy with the help of 'fsolve'; the result is: $\hat{\alpha} = 0.3247$ and $\hat{\beta} = 1.932$.