Difference Equations

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1 Basics

Suppose a sequence of numbers, say a_0, a_1, a_2, a_3, ... is defined by a certain general relationship between, say, three consecutive values of the sequence, e.g.

$$2a_{i+2} + 3a_{i+1} - 5a_i = 7i \tag{1}$$

true for each nonnegative integer i.

An equation of this type is called a DIFFERENCE EQUATION, and our main aim of this article is to explain how to solve it (i.e. find a formula for computing the individual terms of the sequence).

We will deal only with the simplest case of such equations, namely those LINEAR in all the a_i's; furthermore, we allow the a_i's to have only CONSTANT COEFFICIENTS (such as our above example - note that the a_i-terms are usually collected on the left hand side of the equation).

If the right hand side of the equation is zero, it is called HOMOGENEOUS, otherwise (as in our example) the equation is NON-HOMOGENEOUS. We will be able to solve a non-homogeneous equation only when the right hand side is a polynomial in i, further multiplied by a constant raised to the power of i, e.g. $(i^2 - 4) \cdot 2^i$. Note that (1) is of this type - the constant is simply equal to 1.

2 Homogeneous case - Introduction

Having an equation of the

$$2a_{i+2} + 5a_{i+1} - 3a_i = 0 \tag{2}$$

type, we attempt to solve it by assuming that the solution has the following form (hoping that our guess will prove correct):

$$a_i^{\text{trial}} = \lambda^i$$

where λ is a constant whose exact value remains to be established.

Substituting this trial solution into (2) yields

$$2\lambda^{i+2} + 5\lambda^{i+1} - 3\lambda^i = 0$$

or, equivalently (dividing the previous equation by λ^i), the following so called CHARACTERISTIC EQUATION (its LHS is the corresponding CHARACTERISTIC POLYNOMIAL):

$$2\lambda^2 + 5\lambda - 3 = 0$$

The last equation has clearly only two roots, namely $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -3$. This implies that each $a_i = (\frac{1}{2})^i$ and $a_i = (-3)^i$ constitutes a different solution to (2) - we will call them (stretching the usual terminology a bit) BASIC solutions.

One can show that the fully GENERAL SOLUTION of (2) can be built by taking a linear combination of the two basic solutions, thus:

$$a_i^{\text{general}} = A \cdot (\frac{1}{2})^i + B \cdot (-3)^i$$

where A and B are two arbitrary numbers.

To 'fix' the values of A and B and construct a SPECIFIC SOLUTION to (2), we need to be given numerical values of exactly two members of the a_i-sequence. In the case of an equation with two basic solutions, there are two common possibilities:

2.1 Initial conditions

is the name by which we refer to the situation when the first two members of the sequence (usually a_0 and a_1) are given, e.g. $a_0 = -1$ and $a_1 = 2$. This clearly leads to two ordinary linear equations for A and B, namely (in the case of the previous example):

$$\begin{aligned} A + B &= -1 \\ \frac{A}{2} - 3B &= 2 \end{aligned}$$

which can be easily solved to yield: $A = -\frac{2}{7}$ and $B = -\frac{5}{7}$.

The SPECIFIC SOLUTION which meets not only (2) but also the two initial conditions is then:

$$a_i^{\text{specific}} = -\frac{2}{7} \cdot \left(\frac{1}{2}\right)^i - \frac{5}{7} \cdot (-3)^i$$

based on which we can easily find the value of any a_i, e.g. $a_{10} = -\frac{2}{7} \cdot \left(\frac{1}{2}\right)^{10} - \frac{5}{7} \cdot (-3)^{10} = -\frac{21\,595\,063}{512}$.

Note that in this case (of given initial conditions), we can also find a_{10} (or any other specific a_i) by the usual recursive procedure, i.e.

$$\begin{aligned} a_2 &= -\frac{5}{2}a_1 + \frac{3}{2}a_0 = -\frac{13}{2} \\ a_3 &= -\frac{5}{2}a_2 + \frac{3}{2}a_1 = \frac{77}{4} \\ &\vdots \end{aligned}$$

(no formula necessary).

As a second, more interesting example, let us try to find a formula for the determinant of the following tri-diagonal matrix:

$$a_i \equiv \det \begin{bmatrix} \alpha & \beta & 0 & 0 & \dots & 0 \\ \beta & \alpha & \beta & 0 & \dots & 0 \\ 0 & \beta & \alpha & \beta & \dots & 0 \\ 0 & 0 & \beta & \alpha & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \beta \\ 0 & 0 & 0 & \dots & \beta & \alpha \end{bmatrix}$$

where i is the number of rows (and columns).

By expanding it along the first row, one gets:

$$a_i = \alpha \cdot a_{i-1} - \beta^2 \cdot a_{i-2}$$

with $a_1 = \alpha$ and $a_2 = \alpha^2 - \beta^2$.

Clearly,

$$a_{i+2} - \alpha \cdot a_{i+1} + \beta^2 \cdot a_i = 0$$

is an equivalent way of presenting the same set of equations.

The roots of the corresponding characteristic poynomial are

$$\lambda_1 = \frac{\alpha + \sqrt{\alpha^2 - 4\beta^2}}{2}$$

and

$$\lambda_2 = \frac{\alpha - \sqrt{\alpha^2 - 4\beta^2}}{2}$$

The general solution to (**??**) is thus

$$a_i = A \cdot \lambda_1^i + B \cdot \lambda_2^i$$

The initial conditions imply

$$\begin{aligned} A \cdot \lambda_1 + B \cdot \lambda_2 &= \alpha \\ A \cdot \lambda_1^2 + B \cdot \lambda_2^2 &= \alpha^2 - \beta^2 \end{aligned}$$

which yields

$$A = \frac{\lambda_1}{\lambda_1 - \lambda_2} \qquad \text{and} \qquad B = \frac{-\lambda_2}{\lambda_1 - \lambda_2}$$

The final, specific solution is thus

$$a_i = \frac{\lambda_1^{i+1} - \lambda_2^{i+1}}{\lambda_1 - \lambda_2}$$

which can be written in a more explicit form of

$$a_i = \frac{1}{2^i} \cdot \sum_{j=0}^{[i/2]} \binom{i+1}{2j+1} \alpha^{i-2j} (\alpha^2 - 4\beta^2)^j \tag{3}$$

where $[i/2]$ implies the integer part of $i/2$.

2.2 Boundary conditions

We will now return to solving (2), but this time the first and the 'last' value of the sequence are given, e.g. $a_0 = \frac{97}{3}$ and $a_{10} = \frac{629\,857}{32}$.

Similarly to dealing with initial conditions, this leads to two ordinary linear equations for A and B, namely

$$\begin{aligned} A + B &= \frac{97}{3} \\ \frac{A}{2^{10}} + B \cdot (-3)^{10} &= \frac{629\,857}{32} \end{aligned}$$

which yield $A = 32$ and $B = \frac{1}{3}$.

The corresponding specific solution then reads

$$a_i^{\text{specific}} = 32 \cdot \left(\frac{1}{2}\right)^i + \frac{1}{3} \cdot (-3)^i = \left(\frac{1}{2}\right)^{i-5} - (-3)^{i-1}$$

Note that in this case the recursive procedure would not work, and the 'formula' solution is thus the only way to solve the problem.

3 Homogeneous case - Complications

A natural extension of our previous examples is to relate *more than three* consecutive values of the a_i sequence, e.g.

$$a_{i+3} - 3a_{i+2} - a_{i+1} + 3a_i = 0 \tag{4}$$

It's not to difficult to figure out that the corresponding characteristic equation is

$$\lambda^3 - 3\lambda^2 - \lambda + 3 = 0$$

this time having three roots, namely $\lambda_1 = 3$, $\lambda_2 = 1$ and $\lambda_3 = -1$.

The general solution to (4) is thus

$$a_i = A \cdot 3^i + B + C \cdot (-1)^i$$

To find a specific solution, *three* distinct values of the a_i sequence must be given; if these are a_0, a_1 and a_2, we can still call it an initial-value problem (and have the option of solving it recursively), but there is no clear-cut version of 'boundary' conditions.

In general, it is important to realize that there are several equivalent ways of presenting (4), such as

$$a_i - 3a_{i-1} - a_{i-2} + 3a_{i-3} = 0$$

(the only difference being that now one would say: true for $i \geq 3$).

In this context, one should watch out for 'skipped' indices, e.g.

$$a_i - 3a_{i-1} + 2a_{i-3} = 0 \tag{5}$$

whose characteristic polynomial is $\lambda^3 - 3\lambda^2 + 2$ (no λ term!) with roots of 1, $1+\sqrt{3}$ and $1-\sqrt{3}$.

The general solution to (5) is thus

$$a_i = A + B \cdot (1+\sqrt{3})^i + C \cdot (1-\sqrt{3})^i \tag{6}$$

3.1 Double and multiple roots

An obvious complication arises when two (or more) roots of the characteristic polynomial are *equal to each other*, e.g.

$$a_{i+3} - 4a_{i+2} - 3a_{i+1} + 18a_i = 0 \tag{7}$$

whose characteristic polynomial has the following roots: -2, 3 and 3.

Clearly, $(-2)^i$ and 3^i are still two basic solutions of this difference equation, but where is the third?

One can easily verify that, in a case like this, *multiplying* 3^i *by* i creates yet another possible solution to (7) - check it out! The fully *general* solution is then constructed as a linear combination of these:

$$a_i = A \cdot (-2)^i + B \cdot 3^i + C \cdot i \cdot 3^i \tag{8}$$

Converting it to a *specific* solution for a given set of initial values, e.g. $a_0 = 2$, $a_1 = 0$ and $a_2 = -1$ is done in the usual way:

$$\begin{aligned} A + B &= 2 \\ -2A + 3B + 3C &= 0 \\ 4A + 9B + 18C &= -1 \end{aligned}$$

The corresponding solution is: $A = \frac{17}{25}$, $B = \frac{33}{25}$ and $C = -\frac{13}{15}$, which leads to

$$a_i^{\text{spec.}} = \frac{17}{25} \cdot (-2)^i + \frac{11}{25} \cdot 3^{i+1} - \frac{13}{5} \cdot i \cdot 3^{i-1}$$

Similarly, in a case of a triple root (say, equal to -3), we would construct the corresponding three basic solutions by first taking $(-3)^i$, then multiplying this by i, and finally multiplying it by i^2. The general pattern should now be obvious.

Thus, for example, if the characteristic polynomial has the following set of roots: 2, 3, 3, -4, -4 and -4, the general solution is:

$$a_i = A \cdot 2^i + B \cdot 3^i + C \cdot i \cdot 3^i + D \cdot (-4)^i + E \cdot i \cdot (-4)^i + F \cdot i^2 \cdot (-4)^i$$

Note that some textbooks, instead of multiplying a multiple root by i, i^2, i^3, etc. use the following alternate selection: i, $i(i-1)$, $i(i-1)(i-2)$, ... The latter approach has some conceptual and perhaps even computational advantages, but here we prefer using the former. The two schemes are fully equivalent.

3.2 Complex roots

Since this section deals with complex numbers, we need the i symbol to denote the purely imaginary unit. We will thus have to switch from the old a_i to using a_n. Later on, we will return to our regular notation.

As we all know, roots of a polynomial can easily turn out to be complex (coming in complex-conjugate pairs). Formally, our previous solution remains correct, for example:

$$a_{n+2} - 4a_{n+1} + 13a_n = 0 \tag{9}$$

has a characteristic polynomial with roots of $2+3i$ and $2+3i$. We can still write

$$a_n = A \cdot (2+3i)^n + A^* \cdot (2-3i)^n$$

where A and A^* are complex conjugates of each other (to yield a *real* answer), but it is often more convenient to avoid complex numbers and use, instead of $(2+3i)^n$ and $(2-3i)^n$, their real and purely imaginary parts (quite legitimately, as we can always replace any two basic solutions by their linear combinations).

This is achieved more easily by converting $2+3i$ to its *polar representation*, thus:

$$2+3i \equiv \sqrt{13}\left[\cos(\arctan\tfrac{3}{2}) + i\sin(\arctan\tfrac{3}{2})\right]$$

The real and purely imaginary parts of $(2+3i)^n$ are then, respectively:

$$13^{n/2}\cos(n\arctan\tfrac{3}{2})$$

and

$$13^{n/2}\sin(n\arctan\tfrac{3}{2})$$

The general solution to (9) can then be written (avoiding complex numbers) thus:

$$a_n = 13^{n/2}\left[A\cdot\sin(n\arctan\tfrac{3}{2}) + B\cdot\cos(n\arctan\tfrac{3}{2})\right]$$

Luckily, many practical situations usually manage to steer clear of the complex case.

Double and multiple complex roots are dealt with in the standard manner (multiplying the usual basic solution by powers of n). So is the construction of a specific solution.

4 Non-homogeneous case

We now consider difference equations with a *non-zero* right hand side (i.e. having at least one term *not* multiplied by an a_i; such terms can appear on either side of the equation, but we will always transfer them to the RHS).

In general, the RHS can be any expression involving i and a handful of constant parameters, but we will explicitly treat only the case of the RHS having the form of

$$P(i)\cdot\theta^i \tag{10}$$

where $P(i)$ is a *polynomial* in i, and θ is a specific number.

One can show that a *general* solution to a difference equation with a RHS (10) can be built as follows: Take the general solution to the corresponding homogeneous equation, and *add* to it a so called PARTICULAR SOLUTION to the complete (non-homogeneous) equation. Note that the particular solution will have no *arbitrary* constants (the old A, B, C, etc.) - the first part of the solution takes care of these.

To find a particular solution is, for our particular RHS (10), rather easy (at least in principle); all we need to know is that it will have the following form:

$$Q(i) \cdot \theta^i \cdot i^m \tag{11}$$

where $Q(i)$ is a polynomial of the same degree as $P(i)$, but with UNDETERMINED (meaning: yet-to-be-determined) COEFFICIENTS, and m is the multiplicity of θ as a root of the (homogeneous) characteristic polynomial.

Typically, θ is *not* a root of the characteristic polynomial, which means that $m = 0$ and the last factor of (11) disappears. But then, if θ happens to be a simple root, $m = 1$; if it's a double root, $m = 2$, etc.

4.1 Special case of $\theta = 1$

In this case, the RHS is simply a polynomial in i. One has to realize that, *implicitly*, it is still multiplied by 1^i. Thus, to establish the value of m, one has to check whether 1 is a root of the characteristic polynomial (and its multiplicity).

As an example, let us solve Eq. 1. Its characteristic polynomial has the following roots: $-\frac{5}{2}$ and 1. The RHS of (1) is a linear polynomial, which means that $\theta = 1$, whose multiplicity as a root of the characteristic polynomial is $m = 1$. The *form* of the particular solution is then

$$a_i^{\text{part.}} = (q_0 + q_1 i) \cdot i \tag{12}$$

Note that we need a *complete* linear polynomial (i.e. a polynomial with both linear and constant coefficients), even though $P(i)$ had *only* a linear term.

Also, we need to realize that q_0 and q_1 (unlike A and B) are *not* arbitrary numbers; to find the *actual* particular solution, we must first substitute (12) into (1) - this can be done efficiently by Maple - and then *solve* for the *correct* values of q_0 and q_1 by matching the coefficients of all powers of i.

Substituting (12) into the LHS of (1) yields:

$$7q_0 + 11q_1 + 14q_1 i$$

To match this to the RHS of (1), namely to $7i$, we need (matching linear terms):

$$14q_1 = 7$$

which immediately yields $q_1 = \frac{1}{2}$, and (matching absolute terms):

$$7q_0 + 11q_1 = 0$$

which implies that $q_0 = -\frac{11}{14}$.

The particular solution is thus

$$a_i^{\text{part.}} = -\frac{11i}{14} + \frac{i^2}{2}$$

which leads to the following fully general solution of (1):

$$a_i^{\text{gen.}} = A \cdot (-\tfrac{5}{2})^i + B - \frac{11i}{14} + \frac{i^2}{2}$$

Note that only at this point we would be in a position to deal with either initial or boundary conditions (a common mistake is to omit the particular solution when finding A and B).

For example, given that $a_0 = 3$ and $a_1 = 6$, we get

$$\begin{aligned} A + B &= 3 \\ -\frac{5}{2}A + B - \frac{11}{14} + \frac{1}{2} &= 6 \end{aligned}$$

which yields $A = -\frac{46}{49}$ and $B = \frac{193}{49}$, and the following *specific* solution:

$$a_i^{\text{spec.}} = -\frac{46}{49} \cdot (-\tfrac{5}{2})^i + \frac{193}{49} - \frac{11i}{14} + \frac{i^2}{2}$$

4.2 More examples

In this section we concentrate on building particular solutions only (extending them to general solution is quite trivial).

Another frequent special case of (10) arises when the polynomial $P(i)$ is just a constant (of zero degree), e.g.

$$2a_{i+5} - 3a_{i+4} - 24a_{i+3} + 13a_{i+2} + 84a_{i+1} + 36a_i = 5 \cdot 3^i \tag{13}$$

To find the correct form of a particular solution, we only need to know the multiplicity of $\theta = 3$ as a root of the characteristic polynomial

$$2\lambda^5 - 3\lambda^4 - 24\lambda^3 + 13\lambda^2 + 84\lambda + 36 \tag{14}$$

This can be figured out *without* solving the corresponding equation! First, we substitute 3 for λ in (14). Since we get zero, 3 is clearly a root. To find its multiplicity, we repeatedly differentiate (14) with respect to λ, then make the same substitution, till we reach a *non-zero* value. The *order* of the first non-zero derivative yields the multiplicity of the root.

In the case of our example, the first derivative still evaluates to zero, but the second one is equal to 350. This implies that the corresponding multiplicity m is equal to 2. The particular solution will thus have the following *form*:

$$a_i = q \cdot 3^i \cdot i^2 \tag{15}$$

where the correct value of q is yet to be found.

Substituting (15) into the LHS of (13), best done with the help of Maple, yields

$$3150q \cdot 3^i$$

To make this equal to the RHS of (9), $q = \frac{5}{3150} = \frac{1}{630}$.

The particular solution is then

$$a_i^{\text{part.}} = \frac{i^2}{630} \cdot 3^i$$

For our final example, we just replace the RHS of (13) by

$$\frac{3 + i^2}{(-2)^i} \tag{16}$$

i.e. a *quadratic* polynomial, multiplied by θ^i, where $\theta = -\frac{1}{2}$.

First we need to know the multiplicity of $-\frac{1}{2}$ as a root of (14) - the characteristic polynomial has not changed! The corresponding substitution yields 0 for the polynomial itself, and $\frac{441}{8}$ for its first derivative. The corresponding m is thus equal to 1.

The particular solution will have a *form* of

$$\frac{q_0 + q_1 i + q_2 i^2}{(-2)^i} \cdot i$$

When substituted into the LFS of (13), this yields (the amount of algebra is formidable - Maple now becomes indispensable):

$$\left(\frac{-441q_0 - 1105q_1 + 723q_2}{16} - \frac{882q_1 + 315q_2}{16} \cdot i - \frac{1323q_2}{16} \cdot i^2 \right) \cdot \frac{1}{(-2)^i}$$

We can make the last expression equal to (16) by making $q_2 = -\frac{16}{1323}$, $q_1 = \frac{315}{882} \cdot \frac{16}{1323} = \frac{40}{9261}$ and $q_0 = \frac{48+1105q_1-723q_2}{-441} = -\frac{8408}{64827}$.

The particular solution thus reads:

$$a_i^{\text{part.}} = \frac{\frac{1051}{64827} i - \frac{5}{9261} i^2 + \frac{2}{1323} i^3}{(-2)^{i-3}}$$

(we have cancelled out the factor of -8).

4.3 Superposition principle

Finally, when the RHS of a non-homegeneous equation is a *sum* of two or more terms of type (10), it is easy to show that the corresponding particular solution will be the corresponding sum (SUPERPOSITION) of the (two or more) *individual* particular solutions, constructed separately for each term (ignoring the rest).

Thus, replacing the RHS of our previous example (yet one more time) by

$$3 + i^2 - 5 \cdot 2^i \tag{17}$$

we fist have to realize that this does *not* have the form of (10), but can be written as as sum of

$$3 + i^2 \tag{18}$$

and

$$-5 \cdot 2^i \tag{19}$$

which, individually, we do know how to deal with.

The first of these is just a quadratic polynomial (implicitly, $\theta = 1$), and we can also readily verify that 1 is not a root of (14), implying that $m = 0$. The first particular solution will thus have the form of

$$q_0 + q_1 i + q_2 i^2$$

This, substituted into the LHS of (13), yields:

$$108q_2 \cdot i^2 + (108q_1 + 72q_2)i + (108q_0 + 36q_1 - 78q_2)$$

implying that $q_2 = \frac{1}{108}$, $q_1 = -\frac{72}{108^2} = -\frac{1}{162}$ and $q_0 = \frac{3-36q_1+78q_2}{108} = \frac{71}{1944}$.

The corresponding particular solution is

$$a_i^{\text{part}(1)} = \frac{71}{1944} - \frac{i}{162} + \frac{i^2}{108}$$

Similarly, to find a particular solution to match (19), we first have to find the multiplicity of $\theta = 2$ as a root of (14). Substituting 2 for λ again results in a non-zero value, again implying that $m = 0$.

The form of the second particular solution is thus simply

$$q \cdot 2^i$$

This, substituted into the LHS of (13), results in

$$80q \cdot 2^i$$

yielding $q = -\frac{1}{16}$.

Thus,

$$a_i^{\text{part}(2)} = -2^{i-4}$$

The particular solution which solves the whole equation is thus a superposition (sum) of the two, namely:

$$a_i^{\text{part.}} = \frac{71}{1944} - \frac{i}{162} + \frac{i^2}{108} - 2^{i-4}$$

To build the fully general solution, we first need to find all roots of (14). We already know that 3 is a double root and $-\frac{1}{2}$ a simple root; it is not difficult to verify that -2 is also a double root - note that replacing 2^i in (17) by $(-2)^i$ would have made the previous example a lot more difficult!

This implies that

$$a_i^{\text{general}} = A{\cdot}3^i + B{\cdot}i{\cdot}3^i + C{\cdot}(-\tfrac{1}{2})^i + D{\cdot}(-2)^i + E{\cdot}i{\cdot}(-2)^i + \frac{71}{1944} - \frac{i}{162} + \frac{i^2}{108} - 2^{i-4}$$

Only at this point one could start constructing a *specific* solution, based on five given values of a_i. This would lead to a linear set of ordinary equations for A, B, C, D and E (fairly routine). Hopefully, no explicit example is necessary.