```
$HistoryLength = 1;

k = 3;  (* the 1/n degree of approximation *)

Unprotect[Power];   ρ^(l_/;l>3 k+1) := 0   (* truncating powers of ρ *)

ϵ^(l_/;l>2 k) := 0;  (* up to and including 1/n^k terms *)
ns = 5 k + 1;  (* smallest specific n *)

determinant[A_] := Module[{n = Length[A], a = A},
  Do[a[[j]] = a[[j]] - a[[i]] a[[j, i]] / a[[i, i]], {i, n - 1}, {j, i + 1, n}];
  Product[a[[i, i]], {i, n}] // Together]   (* Mathematica' s Det proved inefficient *)

L[i1_, i2_, i3_, i4_, i5_] := D[MGF, {t_1, i1}, {t_2, i2}, {t_3, i3}, {t_4, i4}, {t_5, i5}] /.
  {t_i_ → 0}  (* multivariate moments *)

est = (ρ + ϵ Z_2 - ϵ^2 Z_4 Z_5) / (1 + ϵ Z_1 - ϵ^2 Z_3^2);  (* defining the estimator *)

adj = Series[F[est], {ϵ, 0, 2 k}] - F[est /. ϵ → 0] // Normal;
(* taking a general function of 'est'; also adjusting for ϵ=0 value *)

S = Collect[Z_1 Z_2 Z_3 Z_4 Z_5 {adj, adj^2, adj^3, adj^4}, ϵ, Expand] /. ϵ → 1;
(* ϵ-expanding the first 4 moments,
 the extra Z_1 Z_2 Z_3 Z_4 Z_5 are introduced for technical reasons *)

Do[V0 =
  Table[Which[i == j, 1 + If[i == 1 || i == n, 0, ρ^2], Abs[i - j] == 1, -ρ, True, 0], {i, n}, {j, n}]
   (1 - ρ^2);
V = V0 - t_2 Table[If[Abs[i - j] == 1, 1, 0], {i, n}, {j, n}] -
   2 t_1 Table[If[i == j, 1, 0], {i, n}, {j, n}];
Vio = Table[ρ^Abs[i-j], {i, n}, {j, n}];  (* inverse of V0 *)
T = Table[Which[i == 1, t_3 + t_4, i == n, t_3 + t_5, True, t_3 + t_4 + t_5], {i, n}];
 aux = (V - V0).Vio;
MGF =
  Sqrt[determinant[V0] / determinant[V]] Exp[Nest[Expand[T.Vio - #.aux] &, T.Vio, 2 k - 2].T / 2 // Expand];
MGF = (Series[MGF Exp[-(n - 1) ρ t_2 - n t_1] /. t_i_ → λ t_i / (nn - If[i == 1 || i == 3, 0, 1]),
       {λ, 0, 2 k}] // Normal) /. λ → 1 // Expand;
 (* moment generating function of the Z_i' s; nn represents unevaluated n *)
R[n] = S /. Z_1^i1_. Z_2^i2_. Z_3^i3_. Z_4^i4_. Z_5^i5_. → K[i1 - 1, i2 - 1, i3 - 1, i4 - 1, i5 - 1] /. K → L
 (*resulting set of expected values*), {n, ns, ns + k}]

int = Inverse[Table[Table[i^j, {j, 0, k}], {i, ns, ns + k}]];

res = Table[nn^j, {j, 0, k}].(int.Table[R[ns + j], {j, 0, k}]);
(* converting to a general-n formula *)

res = Apart[res, nn] ;
res = Collect[res /. nn - 1 → 1 / Sum[ϵ^(2 j), {j, 1, k}] /. nn → 1 / ϵ^2, ϵ] ;  (*simplifying*)

F[x_] := x
```

```
moment = Collect[{res[[1]] + F[est /. ϵ → 0], res[[2]] - res[[1]]^2,
    res[[3]] - 3 res[[2]] res[[1]] + 2 res[[1]]^3, res[[4]] - 4 res[[3]] res[[1]] +
     6 res[[2]] res[[1]]^2 - 3 res[[1]]^4}, ϵ, Simplify] /. ϵ^1- → 1 / n^1/2
 (* mean, variance, third and forth central moments *)
```

$$\left\{\frac{-1-3\rho}{n}+\rho+\frac{\rho\,(1+44\rho+68\rho^2+72\rho^3+108\rho^4+100\rho^5+148\rho^6+128\rho^7+188\rho^8+156\rho^9)}{n^3}+\frac{-1+\rho-8\rho^2-8\rho^3-8\rho^4-8\rho^5-8\rho^6-8\rho^7-8\rho^8-8\rho^9-8\rho^{10}}{n^2},\ \frac{1-\rho^2}{n}+\frac{-1+4\rho+15\rho^2}{n^2}+\frac{2\,(-1-16\rho-55\rho^2+14\rho^3+6\rho^4+14\rho^5+6\rho^6+14\rho^7+6\rho^8+14\rho^9+6\rho^{10})}{n^3},\ \frac{6\rho\,(-1+\rho^2)}{n^2}+\frac{4+54\rho-36\rho^2-158\rho^3}{n^3},\ \frac{3\,(-1+\rho^2)^2}{n^2}-\frac{12\,(1-2\rho-14\rho^2+2\rho^3+13\rho^4)}{n^3}\right\}$$

```
Collect[Series[{moment[[3]] / (moment[[2]] n)^3/2, moment[[4]] / (moment[[2]] n)^2},
     {n, Infinity, k}] // Normal, n, Simplify]
  {n^3/2, n^2} // Expand[#, n] &  (* skewness and curtosis *)
```

$$\left\{-\frac{6\rho}{\sqrt{n}\,\sqrt{1-\rho^2}}+\frac{4+45\rho-23\rho^3}{n^{3/2}\,(1-\rho^2)^{3/2}},\ 3+\frac{6-66\rho^2}{n\,(-1+\rho^2)}\right\}$$

```
F[x_] := ArcTanh[x]
```

```
moment =
 Collect[{res[[1]] + F[est /. ϵ → 0], res[[2]] - res[[1]]^2, res[[3]] - 3 res[[2]] res[[1]] +
     2 res[[1]]^3, res[[4]] - 4 res[[3]] res[[1]] + 6 res[[2]] res[[1]]^2 - 3 res[[1]]^4},
   ϵ, Simplify] /. ϵ^1- → 1 / n^1/2  (* same as before *)
```

$$\left\{\frac{1+2\rho}{n\,(-1+\rho^2)}-\frac{2+\rho-3\rho^3}{n^2\,(-1+\rho^2)^2}-\frac{2\,(-3+6\rho+27\rho^2+17\rho^3+12\rho^4-9\rho^5)}{3\,n^3\,(-1+\rho^2)^3}+\mathrm{ArcTanh}[\rho],\ \frac{1}{n^2\,(1-\rho^2)}+\frac{1}{n\,(1-\rho^2)}+\frac{1+48\rho+72\rho^2+36\rho^3+15\rho^4}{3\,n^3\,(-1+\rho^2)^3},\ -\frac{2\,(-1+6\rho+3\rho^2+6\rho^3)}{n^3\,(-1+\rho^2)^3},\ \frac{3}{n^2\,(-1+\rho^2)^2}+\frac{4\,(-2+3\rho^2)}{n^3\,(-1+\rho^2)^3}\right\}$$

```
Collect[Series[{moment[[3]] / (moment[[2]] n)^3/2, moment[[4]] / (moment[[2]] n)^2},
     {n, Infinity, k}] // Normal, n, Simplify]
  {n^3/2, n^2}  // Expand[#, n] & (* skewness and curtosis *)
```

$$\left\{\frac{2\,(-1+6\rho+3\rho^2+6\rho^3)}{n^{3/2}\sqrt{\frac{1}{1-\rho^2}}\,(-1+\rho^2)^2},\ 3+\frac{-2+6\rho^2}{n\,(-1+\rho^2)}\right\}$$

```
Clear[F]
```