```
In[1]:= << DiscreteMath`Combinatorica`
```

```
In[2]:= Unprotect[Times, Power];
```

```
In[3]:= S_{k_} S_{j_} := Module[{n1 = Length[k], n2 = Length[j], k1, k2, p, q, x, y, res = 0, N = 0},
          {k1, k2, n1, n2} = If[n1 < n2, {j, k, n2, n1}, {k, j, n1, n2}];
          p = Sort[Table[NthSubset[i, k2], {i, 0, 2^n2 - 1}]];
          q = Reverse[p];
          Do[x = Permutations[PadRight[p[[i]], n1, N]];
           y = Length[p[[i]]] ! / Length[Permutations[p[[i]]]];
           Do[res = res + y S_{Sort[Join[k1+x[[m]],q[[i]]],Greater]}, {m, 1, Length[x]}], {i, 1, 2^n2}]; res]
```

```
In[4]:= (S_{k_})^m_ := Module[{res = S_k}, Do[res = Expand[S_k res], {j, m - 1}]; Expand[res]]
```

```
In[5]:= Expand[(S_{2,1} + S_{1}) S_{3,1}]
```

Out[5]= $S_{\{3,2\}} + S_{\{4,1\}} + S_{\{4,3\}} + S_{\{5,2\}} + S_{\{3,1,1\}} + S_{\{3,2,2\}} + S_{\{3,3,1\}} + S_{\{4,2,1\}} + S_{\{5,1,1\}} + S_{\{3,2,1,1\}}$

```
In[6]:= S_{k_} S_{j_} := Module[{n1 = Length[k], n2 = Length[j], k1, k2, p, q, x, y, res = 0, N = k[[1]] 0},
          {k1, k2, n1, n2} = If[n1 < n2, {j, k, n2, n1}, {k, j, n1, n2}];
          p = Sort[Table[NthSubset[i, k2], {i, 0, 2^n2 - 1}]];
          q = Reverse[p];
          Do[x = Permutations[PadRight[p[[i]], n1, N]];
           y = Length[p[[i]]] ! / Length[Permutations[p[[i]]]];
           Do[res = res + y S_{Sort[Join[k1+x[[m]],q[[i]]],lexi]}, {m, 1, Length[x]}], {i, 1, 2^n2}]; res]
```

```
In[7]:= lexi[A_, B_] := Module[{N = Length[A]},
          Which[N == 0 || A[[1]] > B[[1]], True,
           A[[1]] < B[[1]], False,
           A[[1]] == B[[1]], lexi[Drop[A, 1], Drop[B, 1]]]]
```

```
In[8]:= Expand[S_{{2,0},{1,0}} S_{{2,1}}^2]
```

Out[8]= $2\,S_{\{\{4,1\},\{3,1\}\}} + S_{\{\{5,2\},\{2,0\}\}} + S_{\{\{6,2\},\{1,0\}\}} + 2\,S_{\{\{3,1\},\{2,1\},\{2,0\}\}} + 2\,S_{\{\{4,1\},\{2,1\},\{1,0\}\}} + S_{\{\{4,2\},\{2,0\},\{1,0\}\}} + S_{\{\{2,1\},\{2,1\},\{2,0\},\{1,0\}\}}$

```
In[9]:= Expand[S_{{1,1,1}}^3]
```

Out[9]= $S_{\{\{3,3,3\}\}} + 3\,S_{\{\{2,2,2\},\{1,1,1\}\}} + S_{\{\{1,1,1\},\{1,1,1\},\{1,1,1\}\}}$

```
In[10]:= EV[A_] := FixedPoint[Expand, A] /. S_{k_} → esip[k] /. esip → ESIP // PowerExpand // Together
```

```
In[11]:= ESIP[k_] := Module[{N = Length[k], res},
           res = Binomial[n, N] N!; Do[res = res MOMENT[k[[i]]], {i, N}]; res]
```

```
In[12]:= MOMENT[k_] := D[Exp[λ (Exp[t] - 1)], {t, k[[1]]}] /. t → 0 // Simplify
```

```
In[13]:= EV[(S_{{1}} / n - λ)^3]
```

$$\text{Out[13]=}\ \frac{\lambda}{n^2}$$

```
In[14]:= MOMENT[k_] :=
          D[Exp[μ1 t1 + μ2 t2 + (σ1^2 t1^2 + σ2^2 t2^2 + 2 ρ σ1 σ2 t1 t2) / 2], {t1, k[[1]]}, {t2, k[[2]]}] /.
            {t1 → 0, t2 → 0} // Simplify
```

```
In[15]:= EV[(S_{{1,0}} / n - μ1) (S_{{0,1}} / n - μ2)]
```

$$\text{Out[15]=}\ \frac{\rho\,\sigma_1\,\sigma_2}{n}$$

```
In[16]:= MOMENT[k_] := D[p / (Exp[-t] - 1 + p), {t, k[[1]]}] /. t → 0 // Simplify

In[17]:= (Series[EV[Series[(n - 1) / (ϵ (S_{{1}} - n / p) + n / p - 1), {ϵ, 0, 6}] // Normal] /.
            {ϵ → 1, n → 1 / v}, {v, 0, 3}] // Simplify) /. v → 1 / n
```

$$\text{Out[17]=}\ p + O\left[\frac{1}{n}\right]^4$$

```
In[18]:= (Series[EV[Series[((n - 1) / (ϵ (S_{{1}} - n / p) + n / p - 1) - p)^2, {ϵ, 0, 6}] // Normal] /.
            {ϵ → 1, n → 1 / v}, {v, 0, 3}] // Simplify) /. v → 1 / n
```

$$\text{Out[18]=}\ -\frac{(-1+p)\,p^2}{n} + 2\,(-1+p)^2\,p^2\left(\frac{1}{n}\right)^2 - 2\,((-1+p)^2\,p^2\,(-2+3\,p))\left(\frac{1}{n}\right)^3 + O\left[\frac{1}{n}\right]^4$$

```
In[19]:= (Series[EV[Series[((n - 1) / (ϵ (S_{{1}} - n / p) + n / p - 1) - p)^3, {ϵ, 0, 6}] // Normal] /.
            {ϵ → 1, n → 1 / v}, {v, 0, 3}] // Simplify) /. v → 1 / n
```

$$\text{Out[19]=}\ (-1+p)\,p^3\,(-4+5\,p)\left(\frac{1}{n}\right)^2 - 4\,((-1+p)\,p^3\,(5-13\,p+8\,p^2))\left(\frac{1}{n}\right)^3 + O\left[\frac{1}{n}\right]^4$$

```
In[20]:= (Series[EV[Series[((n - 1) / (ϵ (S_{{1}} - n / p) + n / p - 1) - p)^4, {ϵ, 0, 6}] // Normal] /.
            {ϵ → 1, n → 1 / v}, {v, 0, 3}] // Simplify) /. v → 1 / n
```

$$\text{Out[20]=}\ 3\,(-1+p)^2\,p^4\left(\frac{1}{n}\right)^2 - (-1+p)\,p^4\,(42-102\,p+61\,p^2)\left(\frac{1}{n}\right)^3 + O\left[\frac{1}{n}\right]^4$$

```
In[21]:= MOMENT[k_] :=
          D[Exp[(t1^2 + t2^2 + 2 ρ t1 t2) / 2], {t1, k[[1]]}, {t2, k[[2]]}] /. {t1 → 0, t2 → 0} // Simplify
```

```
In[22]:= r = (ϵ (S_{{1,1}} / n - ρ) + ρ - ϵ^2 S_{{0,1}} S_{{1,0}} / n^2) /
          Sqrt[(ϵ (S_{{2,0}} / n - 1) + 1 - ϵ^2 S_{{1,0}}^2 / n^2) (ϵ (S_{{0,2}} / n - 1) + 1 - ϵ^2 S_{{0,1}}^2 / n^2)];

In[23]:= μ = Collect[Collect[Series[r, {ϵ, 0, 6}] // Normal, ϵ, EV] /. ϵ → 1, n, Factor] /. n^i_/;i<-3 → 0
```

$$\text{Out[23]=}\ \rho + \frac{(-1+\rho)\,\rho\,(1+\rho)}{2\,n} + \frac{3\,(-1+\rho)\,\rho\,(1+\rho)\,(1+3\,\rho^2)}{8\,n^2} + \frac{3\,(-1+\rho)\,\rho\,(1+\rho)\,(1-2\,\rho^2+25\,\rho^4)}{16\,n^3}$$

```
In[24]:= μ2 = Collect[
            Collect[Series[r^2, {ϵ, 0, 6}] // Normal, ϵ, EV] /. ϵ → 1, n, Factor] /. n^i_/;i<-3 → 0;
```

```
In[25]:= Var = Collect[μ2 - μ^2, n, Factor] /. n^(i_/;i<-3) → 0
```

$$\text{Out[25]= } \frac{(-1+\rho)^2\,(1+\rho)^2}{n} + \frac{(-1+\rho)^2\,(1+\rho)^2\,(2+11\,\rho^2)}{2\,n^2} + \frac{(-1+\rho)^2\,(1+\rho)^2\,(2-2\,\rho^2+75\,\rho^4)}{2\,n^3}$$

```
In[26]:= μ3 = Collect[
            Collect[Series[r^3, {ε, 0, 6}] // Normal, ε, EV] /. ε → 1, n, Factor] /. n^(i_/;i<-3) → 0;
```

```
In[27]:= Collect[Series[(μ3 - 3 μ2 μ + 2 μ^3)/Var^(3/2) /. n → 1 / w, {w, 0, 2}] // Normal,
          w, Simplify[#, ρ^2 < 1] &] /. w → 1 / n
```

$$\text{Out[27]= } -6\sqrt{\frac{1}{n}}\,\rho + \left(\frac{1}{n}\right)^{3/2}\left(12\,\rho - \frac{77\,\rho^3}{2}\right)$$

```
In[28]:= μ4 = Collect[
            Collect[Series[r^4, {ε, 0, 6}] // Normal, ε, EV] /. ε → 1, n, Factor] /. n^(i_/;i<-3) → 0;
```

```
In[29]:= Collect[Series[(μ4 - 4 μ3 μ + 6 μ2 μ^2 - 3 μ^4)/Var^2 /. n → 1 / w, {w, 0, 1}] // Normal,
          w, Simplify[#, ρ^2 < 1] &] /. w → 1 / n
```

$$\text{Out[29]= } 3 + \frac{-6+72\,\rho^2}{n}$$

```
In[30]:= μ = Collect[Collect[Series[ArcTanh[r], {ε, 0, 6}] // Normal, ε, EV] /. ε → 1, n, Factor] /.
          n^(i_/;i<-3) → 0
```

$$\text{Out[30]= } \frac{\rho}{2\,n} + \frac{\rho\,(9+\rho^2)}{8\,n^2} + \frac{3\,\rho\,(13+2\,\rho^2+\rho^4)}{16\,n^3} + \text{ArcTanh}[\rho]$$

```
In[31]:= μ2 = Collect[Collect[Series[ArcTanh[r]^2, {ε, 0, 6}] // Normal, ε, EV] /. ε → 1, n, Factor] /.
           n^(i_/;i<-3) → 0;
```

```
In[32]:= Var = Collect[μ2 - μ^2, n, Factor] /. n^(i_/;i<-3) → 0
```

$$\text{Out[32]= } \frac{1}{n} + \frac{6-\rho^2}{2\,n^2} + \frac{52-12\,\rho^2-3\,\rho^4}{6\,n^3}$$

```
In[33]:= μ3 = Collect[Collect[Series[ArcTanh[r]^3, {ε, 0, 6}] // Normal, ε, EV] /. ε → 1, n, Factor] /.
           n^(i_/;i<-3) → 0;
```

```
In[34]:= Collect[Series[(μ3 - 3 μ2 μ + 2 μ^3)/Var^(3/2) /. n → 1 / w, {w, 0, 2}] // Normal,
          w, Simplify[#, ρ^2 < 1] &] /. w → 1 / n
```

$$\text{Out[34]= } \left(\frac{1}{n}\right)^{3/2}\rho^3$$

```
In[35]:= μ4 = Collect[Collect[Series[ArcTanh[r]^4, {ε, 0, 6}] // Normal, ε, EV] /. ε → 1, n, Factor] /.
           n^(i_/;i<-3) → 0;
```

In[36]:= $\texttt{Collect}\left[\texttt{Series}\left[\frac{\mu_4 - 4\,\mu_3\,\mu + 6\,\mu_2\,\mu^2 - 3\,\mu^4}{\texttt{Var}^2}\ \texttt{/. n} \to \texttt{1 / w, \{w, 0, 1\}}\right]\ \texttt{// Normal,}\right.$
$\left.\texttt{w, Simplify[\#, }\rho^2 < 1\texttt{] \&}\right]\ \texttt{/. w} \to \texttt{1 / n}$

Out[36]= $3 + \frac{2}{n}$