STIRLING'S FORMULA

Bernoulli polynomials are defined recursively in the following manner:

We start with

$$B_1(y) \equiv y$$

Then, we introduce $B_2(y)$ by

$$B_2(y) = 2\int_0^y B_1(u)du + C$$

where C is chosen so that

$$\int_{-1/2}^{1/2} B_2(y)dy = 0$$

yielding

$$B_2(y) = y^2 - \frac{1}{12}$$

Note that $B_2(y)$ is an *even* function of y, implying that (i) $B_2(-\frac{1}{2}) = B_2(\frac{1}{2})$, and (ii) it integrates to 0 both in the $(-\frac{1}{2}, 0)$ and the $(0, \frac{1}{2})$ range.

The next polynomial is simply

$$B_3(y) = 3\int_0^y B_2(u)du = y^3 - \frac{y}{4}$$

Note that (i) it is a *odd* function of y, and (ii) $B_3(-\frac{1}{2}) = B_3(\frac{1}{2}) = 0$, due to the second property of the $B_2(y)$ function.

This is followed by

$$B_4(y) = 4\int_0^y B_3(u)du + C = y^4 - \frac{y^2}{2} + \frac{7}{240}$$

again, an even function which integrates (individually, on each half of the $-\frac{1}{2}$ to $\frac{1}{2}$ range) to zero, and

$$B_5(y) = 5\int_0^y B_4(u)du = y^5 - \frac{5}{6}y^3 + \frac{7}{48}y$$

(an odd function equal to zero at $-\frac{1}{2}$ and $\frac{1}{2}$).

In this manner, we can continue indefinitely. The main properties of the resulting polynomials are

$$\begin{aligned} B_n'(y) &= n \cdot B_{n-1}(y) \\ B_{2k}(-\tfrac{1}{2}) &= B_{2k}(\tfrac{1}{2}) \equiv b_k \\ B_{2k-1}(-\tfrac{1}{2}) &= B_{2k-1}(\tfrac{1}{2}) = 0 \end{aligned}$$

for every positive n and k (with the understanding that $B_0(y) \equiv 1$, to meet the first identity with $n = 1$). Note that, on the second line, we introduced a new symbol b_k for the corresponding Bernoulli number (one can easily evaluate these to equal to $\frac{1}{6}$, $-\frac{1}{30}$, $\frac{1}{42}$, $-\frac{1}{30}$, $\frac{5}{66}$, ... for $k = 1, 2, 3, 4, 5, ...$).

Exponential generating function of these is defined as

$$G(y,t) \equiv \sum_{n=1}^{\infty} \frac{B_n(y)\cdot t^n}{n!} \tag{1}$$

and can be obtained by first differentiating each side of (1) with respect to y, getting

$$\frac{dG(y,t)}{dy} = \sum_{n=1}^{\infty} \frac{n\cdot B_{n-1}(y)\cdot t^n}{n!} = t\sum_{m=0}^{\infty} \frac{B_m(y)\cdot t^m}{m!} = t\cdot G(y,t)+t$$

Solving this ODE yields

$$G(y,t) = c(t)\cdot \exp(t\cdot y) - 1$$

Since integrating (1) over y from $-\frac{1}{2}$ to $\frac{1}{2}$ returns zero (each $B_n(y)$ integrates to zero), the same must hold for the last solution, namely

$$c(t)\int_{-1/2}^{1/2} \exp(t\cdot y)dy - 1 = c(t)\cdot \frac{\exp(\frac{t}{2})-\exp(-\frac{t}{2})}{t} - 1 = 0$$

Solving for $c(t)$, the final formula for the EGF is thus

$$G(y,t) = \frac{t\cdot \exp(t\cdot y)}{\exp(\frac{t}{2})-\exp(-\frac{t}{2})} - 1 \tag{2}$$

Based on this, we get two important results, namely

1. By setting $y=-\frac{1}{2}$ we get

$$\frac{t\cdot \exp(-\frac{t}{2})}{\exp(\frac{t}{2})-\exp(-\frac{t}{2})} - 1 = \frac{t}{\exp(t)-1} - 1$$

   which is the EGF of Bernoulli *numbers* (if we consider coefficients of only the positive *even* powers of t).

2. Replacing y by $\frac{1}{2}$ and t by $2i\cdot u$ (i is the purely imaginary unit) on the RHS of (**??**) and (1), we get

$$\begin{aligned} i\cdot u + \sum_{k=1}^{\infty} \frac{b_k\cdot(2i\cdot u)^{2k}}{(2k)!} &= i\cdot u + \sum_{k=1}^{\infty} \frac{(-1)^k b_k\cdot(2u)^{2k}}{(2k)!} = \\ \frac{2i\cdot u\cdot \exp(i\cdot u)}{\exp(i\cdot u)-\exp(-i\cdot u)} - 1 &= \frac{u\,(\cos u + i\cdot \sin u)}{\sin u} - 1 \end{aligned}$$

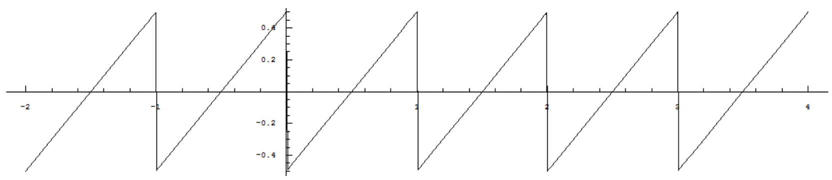
   Solving for the infinite sum yields

$$\sum_{k=1}^{\infty} \frac{(-1)^k b_k\cdot(2u)^{2k}}{(2k)!} = u\cdot \cot u - 1$$

Bernoulli functions Technically, Bernoulli polynomials, denoted $\hat{B}_n(x)$, are defined on the $(0,1)$ interval, instead of our $(-\frac{1}{2},\frac{1}{2})$; the n^{th} of them is thus (using a somehow unconventional notation):

$$\hat{B}_n(x) \equiv B_n(x - \tfrac{1}{2})$$

These can be converted into what we call Bernoulli *functions* (we will use the same $\hat{B}_n(x)$ notation for these) by extending them, *periodically* (meaning $\hat{B}_n(x) \equiv \hat{B}_n(x-1)$) throughout the real axis. For example, $\hat{B}_n(x)$ looks as follows:



Euler-Maclaurin formula is, effectively, a 'correction' to the well known trapezoidal rule, namely:

$$\int_m^\ell f(x)dx \simeq \sum_{j=m}^{\ell} f(j) - \frac{f(m)+f(\ell)}{2} - \sum_{j=1}^{\infty} \frac{b_j}{(2j)!}\left(f^{(2j-1)}(\ell) - f^{(2j-1)}(m)\right) \quad (3)$$

where m and ℓ are two integers, $f^{(2j-1)}$ indicates the corresponding derivative, and the upper limit of the last summation is rather symbolic (one would normally use only a handful of terms). The proof is simple:

$$\begin{aligned}
\int_m^\ell f(x)dx &= \sum_{j=m}^{\ell-1}\int_j^{j+1} \hat{B}_1'(x)\cdot f(x)dx = \sum_{j=m}^{\ell-1} \hat{B}_1(x)\cdot f(x)\Big|_{x=j}^{j+1} - \sum_{j=m}^{\ell-1}\int_j^{j+1} \hat{B}_1(x)\cdot f'(x)dx \\
&= \sum_{j=m}^{\ell-1} \frac{f(j+1)+f(j)}{2} - \int_m^\ell \frac{\hat{B}_2'(x)}{2}\cdot f'(x)dx \\
&= \sum_{j=m}^{\ell} f(j) - \frac{f(m)+f(\ell)}{2} - \frac{\hat{B}_2(x)}{2}\cdot f'(x)\Big|_{x=m}^{\ell} + \int_m^\ell \frac{\hat{B}_2(x)}{2}\cdot f''(x)dx \\
&= \sum_{j=m}^{\ell} f(j) - \frac{f(m)+f(\ell)}{2} - b_1\cdot\frac{f'(\ell)-f'(m)}{2} + \int_m^\ell \frac{\hat{B}_3'(x)}{3!}\cdot f''(x)dx
\end{aligned}$$

The last term is equal to

$$\begin{aligned}
& -\int_m^\ell \frac{\hat{B}_3(x)}{3!} \cdot f'''(x)dx \\
= & -\int_m^\ell \frac{\hat{B}_4'(x)}{4!} \cdot f'''(x)dx \\
= & -\frac{\hat{B}_4(x)}{4!} \cdot f'''(x)\bigg|_{x=m}^{\ell} + \int_m^\ell \frac{\hat{B}_4(x)}{4!} \cdot f^{(4)}(x)dx \\
= & -b_2 \cdot \frac{f'''(\ell) - f'''(m)}{4!} + \int_m^\ell \frac{\hat{B}_5'(x)}{5!} \cdot f^{(4)}(x)dx
\end{aligned}$$

whose the last term equals

$$\begin{aligned}
& -\int_m^\ell \frac{\hat{B}_5(x)}{5!} \cdot f'^{(5)}(x)dx \\
= & -\int_m^\ell \frac{\hat{B}_6'(x)}{6!} \cdot f^{(5)}(x)dx \\
= & -\frac{\hat{B}_6(x)}{6!} \cdot f^{(5)}(x)\bigg|_{x=m}^{\ell} + \int_m^\ell \frac{\hat{B}_6(x)}{6!} \cdot f^{(6)}(x)dx \\
= & -b_3 \cdot \frac{f^{(5)}(\ell) - f^{(5)}(m)}{6!} + \int_m^\ell \frac{\hat{B}_7'(x)}{7!} \cdot f^{(6)}(x)dx
\end{aligned}$$

etc.

Note that, since $\hat{B}_1(x)$ has a discontinuity at each integer, the corresponding integration had to be broken down into continuous segments (no longer necessary for the remaining Bernoulli functions, which are all continuous).

An important special case of (3) lets ℓ tend to ∞ and assumes that the corresponding limit of all $f^{(2j-1)}(\ell)$ derivatives (including $f(\ell)$ itself) is 0. One then gets

$$\int_m^\infty f(x)dx \simeq \sum_{j=m}^{\infty} f(j) - \frac{f(m)}{2} + \sum_{j=1}^{\infty} \frac{b_j}{(2j)!} f^{(2j-1)}(m)$$

Stirling's formula: When $f(x) = \ln(x)$, $m = 1$, and ℓ is replaced by more common (in this context) n, (3) implies

$$n \cdot \ln n - n + 1 \simeq \ln n! - \frac{\ln n}{2} - \sum_{j=1}^{\infty} \frac{b_j}{(2j)(2j-1)} \left(\frac{1}{n^{2j-1}} - 1 \right)$$

since the $2j-1^{th}$ derivative of $\ln x$ is $(2j-2)!x^{-2j+1}$. Actually, in this form, the formula is meaningless because the last (infinite) summation is *not* convergent

!! We must re-write it (rearranging its terms in the process) as

$$\ln n! = n \cdot \ln n - n + \frac{\ln n}{2} + \sum_{j=1}^{k} \frac{b_j}{(2j)(2j-1)n^{2j-1}} + c_k + R_{n,k}$$

where we have made the summation finite, added the corresponding error term $R_{n,k}$, and replaced

$$1 - \sum_{j=1}^{\infty} \frac{b_j}{(2j)(2j-1)}$$

by c_k. It is easy to see that, as n increases, $c_k + R_{n,k}$ will tend to a finite limit which is independent of k and remains to be established. The best we can do in terms of finding a good approximation for $\ln n!$ is to replace $c_k + R_{n,k}$ by this limit, equal to

$$\lim_{n\to\infty} \left(\ln n! - n \cdot \ln n + n - \frac{\ln n}{2} \right) \tag{4}$$

as all terms of the j summation tend to zero.

To do this, we recall that

$$n! = \int_0^{\infty} x^n \exp(-x) dx \tag{5}$$

Expanding ln of the integrand in x at n yields

$$\ln[x^n \exp(-x)] \simeq \ln[n^n \exp(-n)] - \frac{(x-n)^2}{2n} + ...$$

implying that

$$x^n \exp(-x) \simeq n^n \exp(-n) \cdot \exp\left(-\tfrac{(x-n)^2}{2n}\right)$$

One can refine this approximation by further expanding the ratio of the LHS to the RHS, getting

$$x^n \exp(-x) \simeq n^n \exp(-n) \cdot \exp\left(-\frac{(x-n)^2}{2n}\right) \cdot \left(1 + \frac{(x-n)^3}{3n^2} - \frac{(x-n)^4}{4n^3} + \frac{(x-n)^5}{5n^4} + ...\right)$$

but this will not affect the resulting $n \to \infty$ limit, as one can easily verify.

Using this formula,

$$\ln n! \simeq n \ln n - n + \ln \left[\int_0^{\infty} \exp\left(-\frac{(x-n)^2}{2n}\right) dx \right]$$

When series expanding the last term in n at infinity we get

$$\ln \sqrt{2n\pi} + ...$$

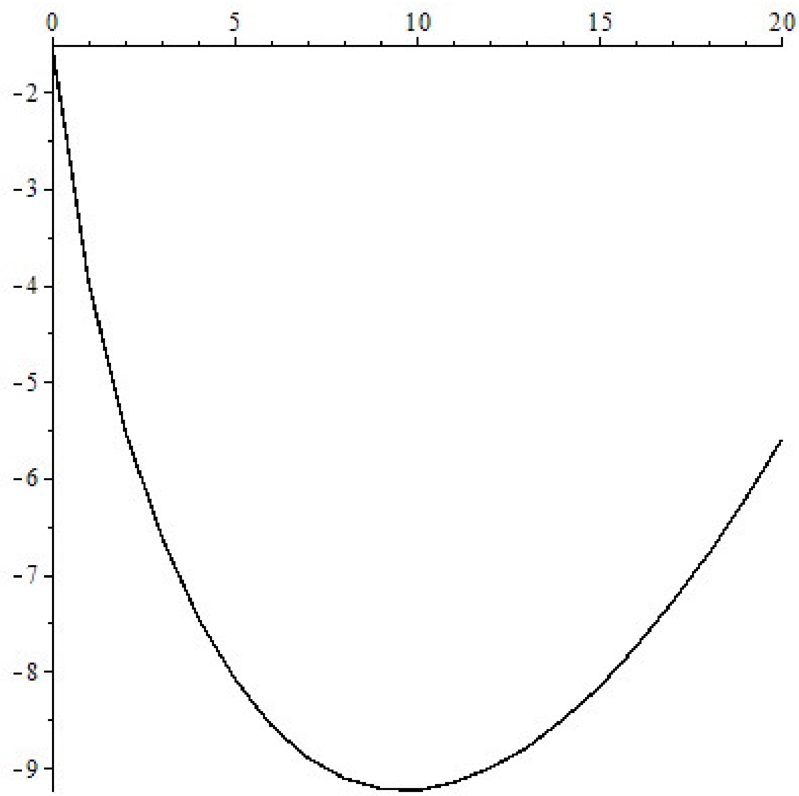
This clearly shows that (4) is equal to

$$\ln \sqrt{2\pi}$$

The final approximation is thus

$$\begin{aligned}\ln n! &\simeq n \cdot \ln n - n + \frac{\ln(2\pi n)}{2} + \sum_{j=1}^{k} \frac{b_j}{(2j)(2j-1)n^{2j-1}} \\ &= n \cdot \ln n - n + \frac{\ln(2\pi n)}{2} + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} + \frac{1}{1188n^9} + ...\end{aligned}$$

The following graph displays the log 10 of absolute error of this formula (plotted against the value of k) when approximating $\ln 3!$



The asymptotic nature of the approximation is quite obvious.