Simulating n-link chain sliding off a desktop

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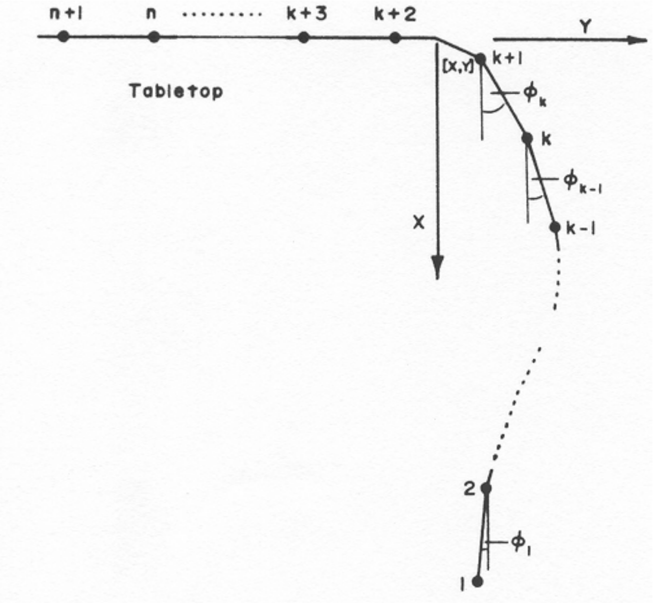
Abstract

The main purpose of this article is to spell out, in detail, the algorithm for simulating a chain sliding off a table top, thus complementing [1]. This should enable the reader to write the corresponding program, in a computing language of their choice.

Consider a chain consisting of $n+1$ point-like particles of the same mass (equal to 1, by a choice of units), connected by n massless, friction-less, unstretchable links of equal length (also equal to 1). The chain is laid on a table top, straight and perpendicular to the edge. Then, the end particle is pulled (together with the rest of the chain), gently, over the edge of the table. This causes the chain, due to gravity (also of size 1) to start sliding down.

We will assume that $k+1$ particles have already left the table top, and that their position is defined by k angles ϕ_1, ϕ_2, ..., by which the corresponding link deviates from the vertical, and by the coordinates of the $k+1^{st}$ particle, denoted X and Y respectively (collectively known as the GENERALIZED COORDINATES).



The x coordinate of the i^{th} particle is thus

$$x_i = \begin{cases} X + \sum_{j=i}^{k} \cos\phi_j & 1 \le i \le k+1 \\ 0 & k+2 \le i \le n+1 \end{cases}$$

Similarly, the y coordinates equal

$$y_i = \begin{cases} Y + \sum_{j=i}^{k} \sin\phi_j & 1 \le i \le k+1 \\ \sqrt{X^2+Y^2} + k + 1 - i & k+2 \le i \le n+1 \end{cases}$$

The corresponding Lagrangian is

$$\begin{aligned} L &= \sum_{i=1}^{n+1} \left(\frac{\dot{x}_i^2 + \dot{y}_i^2}{2} + x_i \right) = \frac{n-k}{2} \left(\overset{\bullet}{\sqrt{X^2+Y^2}} \right)^2 + \frac{k+1}{2} (\dot{X}^2 + \dot{Y}^2) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{k} \min(i,j) \dot{\phi}_i \dot{\phi}_j \cos(\phi_i - \phi_j) + \sum_{i=1}^{k} i \ \dot{\phi}_i (\dot{Y} \cos\phi_i - \dot{X} \sin\phi_i) \\ &\quad + (k+1) \ X + \sum_{i=1}^{k} i \ \cos\phi_i \end{aligned} \tag{1}$$

where $\overset{\bullet}{\sqrt{X^2+Y^2}} = \frac{\dot{X}X + \dot{Y}Y}{\sqrt{X^2+Y^2}}$.

This leads to the following set of $k+2$ differential equations for for ϕ_1, ϕ_2, ... ϕ_k, X and Y:

$$\begin{bmatrix} \min(i,j)\cos(\phi_i - \phi_j) & -i \sin\phi_i & i \cos\phi_i \\ -j \sin\phi_j & (n-k)\frac{X^2}{X^2+Y^2} + k + 1 & (n-k)\frac{XY}{X^2+Y^2} \\ j \cos\phi_j & (n-k)\frac{XY}{X^2+Y^2} & (n-k)\frac{Y^2}{X^2+Y^2} + k + 1 \end{bmatrix} \cdot \begin{bmatrix} \ddot{\phi}_i \\ \ddot{X} \\ \ddot{Y} \end{bmatrix}$$

$$= \begin{bmatrix} -\sum_{j=1}^{k} \dot{\phi}_j^2 \min(i,j) \sin(\phi_i - \phi_j) - i \sin\phi_i \\ \sum_{j=1}^{k} \dot{\phi}_j^2 \cos\phi_j + (k+1) - (n-k) \ X \ \left(\overset{\bullet}{\arctan \frac{Y}{X}} \right)^2 \\ \sum_{j=1}^{k} \dot{\phi}_j^2 \sin\phi_j - (n-k) \ Y \ \left(\overset{\bullet}{\arctan \frac{Y}{X}} \right)^2 \end{bmatrix} \tag{2}$$

where $1 \le i, \ j \le k$ and $\overset{\bullet}{\arctan \frac{Y}{X}} = \frac{\dot{Y}X - X\dot{Y}}{X^2+Y^2}$. When both X and Y are equal to zero, $\frac{X}{\sqrt{X^2+Y^2}}$ and $\frac{Y}{\sqrt{X^2+Y^2}}$ have (in this limit) the values of $\frac{\dot{X}}{\sqrt{\dot{X}^2+\dot{Y}^2}}$ and $\frac{\dot{Y}}{\sqrt{\dot{X}^2+\dot{Y}^2}}$ respectively (always well defined), and $\frac{\dot{Y}X - X\dot{Y}}{X^2+Y^2}$ remains finite.

The above equations can be numerically integrated until $\sqrt{X^2+Y^2}$ reaches the value of 1, meaning that the $k+2^{nd}$ particle is just about to leave the table top.

What we need to do then is:

1. Introduce ϕ_{k+1} and d to be the angle and length of the $k+1^{st}$ link (setting their initial values to $\arctan\frac{Y}{X}$ and 1 respectively), and X_N and Y_N to be the coordinates of particle $k+2$ (both having the initial values of zero). Note that the initial value of $\dot{\phi}_{k+1}$, $\dot{d}$, $\dot{X}_N$ and $\dot{Y}_N$ are

   $$\overset{\bullet}{\arctan\frac{Y}{X}} - X\overset{\bullet}{\sqrt{X^2+Y^2}}$$

   $$(1-Y)\overset{\bullet}{\sqrt{X^2+Y^2}}$$

   $$0$$

   $$\overset{\bullet}{\sqrt{X^2+Y^2}}$$

   respectively.

2. Allowing d to remain a function of time, apply a strong constant force F along this link, pulling particles $k+1$ and $k+2$ together, until $\dot{d}$ becomes zero (the force will than cease). This can be achieved by the following steps:

   (a) Set up the appropriate Lagrangian, namely

   $$\begin{aligned} L &+ \frac{k+1}{2}\ \dot{d}^2 + \dot{d}\ \left[(k+1)\dot{X}_N\cos\phi_{k+1} + (k+1)\dot{Y}_N\sin\phi_{k+1} - \sum_{i=1}^{k} i\ \dot{\phi}_i\sin(\phi_i-\phi_{k+1})\right] \\ &+ \frac{k+1}{2}\ d^2\dot{\phi}_{k+1}^2 + d\ \dot{\phi}_{k+1}\left[(k+1)\dot{Y}_N\cos\phi_{k+1} - (k+1)\dot{X}_N\sin\phi_{k+1} + \sum_{i=1}^{k} i\ \dot{\phi}_i\cos(\phi_i-\phi_{k+1})\right] \\ &+ d\ \left[(k+1)\cos\phi_{k+1} - F\right] - \frac{1}{2}\left(\overset{\bullet}{\sqrt{X_N^2+Y_N^2}}\right)^2 + \frac{1}{2}(\dot{X}_N^2+\dot{Y}_N^2) + X_N \end{aligned} \tag{3}$$

   where L is the old (1) with X and Y replaced by X_N and Y_N respectively.

   (b) Derive the corresponding set of differential equations for ϕ_1, ϕ_2, ... ϕ_{k+1}, d, X_N and Y_N (the new generalized coordinates).

   (c) Instead of the old time t, introduce $\tau \equiv F\cdot t$ (the corresponding derivatives will be denoted by a prime, rather than a dot).

   (d) Correspondingly replace $\dot{q}$ by $F\ q'$ and $\ddot{q}$ by $F^2 q''$ (where q represents each of our generalized coordinates),

   (e) Introduce $\Psi_i \equiv F\cdot\phi_i' = \dot{\phi}_i$, for $1\le i\le k+1$, $\Omega\equiv F\cdot d' = \dot{d}$, $\Gamma\equiv F\cdot X_N' = \dot{X}_N$. and $\Lambda\equiv F\cdot Y_N' = \dot{Y}_N$.

   (f) Take the $F\to\infty$ limit (i.e. keep only terms proportional to F). Realize that, in this limit, all generalized coordinates are 'frozen' in time, i.e. don't change their values, and only their time derivatives (momenta) change (in real time, instantaneously). The value of d will thus remain equal to 1.

(g) Numerically integrate the corresponding set of differential equations for Ψ_1, Ψ_2, ... Ψ_{k+1}, Γ, Λ and Ω, namely

$$\begin{bmatrix} \min(i,j)\cos(\phi_i-\phi_j) & -i\ \sin\phi_i & i\ \cos\phi_i & -i\ \sin(\phi_i-\phi_{k+1}) \\ -j\ \sin\phi_j & (n-k-1)\frac{X_N^2}{X_N^2+Y_N^2}+k+2 & (n-k-1)\frac{X_N Y_N}{X_N^2+Y_N^2} & (k+1)\cos\phi_{k+1} \\ j\ \cos\phi_j & (n-k-1)\frac{X_N Y_N}{X_N^2+Y_N^2} & (n-k-1)\frac{Y_N^2}{X_N^2+Y_N^2}+k+2 & (k+1)\sin\phi_{k+1} \\ -j\ \sin(\phi_j-\phi_{k+1}) & (k+1)\cos\phi_{k+1} & (k+1)\sin\phi_{k+1} & k+1 \end{bmatrix} \cdot \begin{bmatrix} \Psi_i' \\ \Gamma' \\ \Lambda' \\ \Omega' \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad (4)$$

(where $1 \le i, j \le k+1$) until $\Omega = 0$. The tricky part is computing the values of X_N and Y_N (needed in the central 2×2 block). Initially, they are both equal to zero and need to be replaced by the initial values of $\dot{X}_N$ and $\dot{Y}_N$. After that, they become two extra dependent variables of the τ integration, with $X_N' = \frac{\Gamma}{F}$ and $Y_N' = \frac{\Lambda}{F}$ (for this purpose equivalent to $X_N' = \Gamma$ and $Y_N' = \Lambda$, since F cancels out). This means that the actual increase of X_N and Y_N is only 'infinitesimal' and, in the $F \to \infty$ limit, both X_N and Y_N retain their zero values.

This yields the initial values of all momenta for the next step.

(h) Increase the value of k by one and perform numerical integration of (2).

Starting with $X = 0$, $Y = 0.0001$, $\dot{X} = 0$, $\dot{Y} = 0$ and $k = 0$, loop through this procedure until k reaches the value of n.

References

[1] *Chain sliding off a table.* Am. J. Phys. **61** (3), March 1993