# Generating functions in combinatorics

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There are two basic issues in Combinatorics; here we give a brief introduction to each.

## 1 Selecting $r$ objects out of $n$

This is ambiguous unless we specify whether (or not)

- we can select the same object more than once (as many times as we like),
- the order in which we make the selection makes a difference.

Accordingly, we have to discuss four distinct cases.

### 1.1 Selecting $r$ different objects, order irrelevant

Keeping $n$ fixed, there is a simple way of generating all the answers (for $0 \le r \le n$). How we can always do that should be obvious from the following example.

**Example 1** *When* $n = 5$, *expanding*

$$\begin{aligned}&(1+a)(1+b)(1+c)(1+d)(1+e) = \qquad (1)\\ &1+a+b+c+d+e+ab+ac+ad+bc+bd+cd+ae+be+ce+de+abc+abd+acd\\ &+bcd+abcd+abe+ace+ade+bce+bde+cde+abce+abde+acde+bcde+abcde\end{aligned}$$

*indicates that there is only* 1 *way of selecting no object (or, selecting all five),* 5 *ways of selecting one object (or* 4 *objects - equivalent to deciding which object* not *to select), and* 10 *ways of selecting two (or, equivalently three) objects. Not that, since the order of selection does not matter, we may as well arrange the objects alphabetically (in general, this prevents us from listing a selection, such as* $abc$, *more than once, e.g.* $acb$*).*

Note that when replacing each of the 5 distinct symbols by $x$, we get the so called (ordinary) GENERATING FUNCTION for the corresponding *number* of selections (coefficients of $x^r$), thus

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

In general, we know (from Binomial Theorem) that the coefficient of $x^r$ in the expansion of $(1+x)^n$ is the binomial coefficient

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad (2)$$

## 1.2 Repetition allowed, order irrelevant

This time, we have to replace each factor of the $(1+a)$ type in (1) by

$$1 + a + a^2 + a^3 + ... = \frac{1}{1-a}$$

**Example 2** *For $n = 5$ and $r = 3$ we get*

$$\frac{1}{(1-a)(1-b)(1-c)(1-d)(1-e)} \simeq ... +$$
$$a^3 + b^3 + c^3 + d^3 + e^3 + abc + abd + acd + bcd + abe + ace + ade + bce + bde + cde$$
$$+ae^2 + be^2 + ce^2 + de^2 + ab^2 + a^2b + ac^2 + a^2c + ad^2 + bc^2 + a^2d + b^2c + bd^2$$
$$+b^2d + cd^2 + c^2d + a^2e + b^2e + c^2e + d^2e + ...$$

*where $\simeq$ indicates a multivariate Taylor expansion (in terms of $a$, $b$, $c$, $d$ and $e$) and, on the RHS, we list only the* ***cubic*** *terms ($a^2c$ means: select two $a$s and one $c$).*

The corresponding generating function yields

$$\frac{1}{(1-x)^5} \simeq 1 + 5x + 15x^2 + 35x^3 + ...$$

implying that there are 5 ways of selecting one letter, 15 ways of selecting two and 35 ways (listed above) to select three (not necessarily distinct) letters, etc.

This means that a general formula for the number of ways to select $r$ objects out of $n$ (in this particular manner) is the coefficient of $x^r$ in the Taylor (generalized Binomial) expansion of $(1-x)^{-n}$, namely

$$(-1)^r \binom{-n}{r} = \frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r} \tag{3}$$

This is also the number of way of permuting $r$ circles and $n-1$ bars (each such permutation yielding a unique selections of letters, e.g. $| \bigcirc | \bigcirc\bigcirc ||$ corresponds to $a^0b^1c^2d^0e^0$, etc.).

## 1.3 Repetition allowed, order matters

This time we have to use EXPONENTIAL generating functions (explained below).

**Example 3** *With $n = 4$ and $r = 3$ we now get*

$$\exp(a+b+c+d) \simeq ... +$$
$$\frac{\begin{array}{c} 6abc + 6abd + 6acd + 6bcd + a^3 + b^3 + c^3 + d^3 + 3ab^2 \\ +3a^2b + 3ac^2 + 3a^2c + 3ad^2 + 3bc^2 + 3a^2d + 3b^2c + 3bd^2 + 3b^2d + 3cd^2 + 3c^2d \end{array}}{3!} + ...$$

*where each coefficient is now divided by the corresponding $r!$ . Furthermore, $6abc$ is to be interpreted as: $abc$ with all its $\binom{3}{1,1,1}$ permutations, $3a^2b$ as $aab$ with all its $\binom{3}{2,1,0}$ permutations, etc. Note that we would be able to get explicitly all such permutations if multiplication were* ***non****-commutative.*

Expanding

$$\exp(4x) = 1 + 4 \cdot \frac{x}{1!} + 4^2 \cdot \frac{x^2}{2!} + 4^3 \cdot \frac{x^3}{3!} + ..$$

tells us that there are 4 ways of doing this when selecting one letter (out of 4), $4^2$ ways when selecting two, $4^3$ when selecting three, etc.

The general formula for the number of possible ways to do this (creating distinct $r$-letter 'words' based on $n$-letter 'alphabet') is

$$n^r \tag{4}$$

This can be derived more directly by considering how to fill in $r$ 'blanks', each with a choice of one of $n$ letters.

## 1.4 Without duplication, order matters

The generating function is the same as in the first case, namely $(1+x)^n$, except now it has to be interpreted as *exponential* (rather than ordinary) GF.

**Example 4** *Going back to* $n = 5$ *results in*

$$(1+a)(1+b)(1+c)(1+d)(1+e) =$$
$$1 + a + b + c + d + e + \frac{2ab + 2ac + 2ad + 2bc + 2bd + 2cd + 2ae + 2be + 2ce + 2de}{2!} +$$
$$\frac{\begin{matrix}6abc + 6abd + 6acd + 6bcd + 6abcd + 6abe + 6ace + 6ade + 6bce + \\ 6bde + 6cde + 6abce + 6abde + 6acde + 6bcde + 6abcde\end{matrix}}{3!}$$

*where again,* $2ab$ *implies* $ab$ *and* $ba$, $6abc$ *means* $abc$ *with all its* $3!$ *permutations, etc.*

Correspondingly

$$(1+x)^5 \simeq 1 + 5x + 20 \cdot \frac{x^2}{2} + 60 \cdot \frac{x^3}{3!} + 120 \cdot \frac{x^4}{4!} + 120 \cdot \frac{x^5}{5!}$$

implies that there are 5 such 'words' consisting one letter, 20 having two, 60 having three (*distinct*) letters, etc.

The general formula for the number of ways to do this is clearly

$$\binom{n}{r} \cdot r! = \frac{n!}{(n-r)!} \tag{5}$$

Again, one can derive it easily by filling in $r$ blanks by *distinct* letters; thus, we have $n$ choices to fill the first blank, $n-1$ for the second one, $n-2$ for the next, etc. Since these follow the multiplication principle, we get

$$n(n-1)(n-2)...(n-r+1)$$

which agrees with the previous formula.

Now, we turn to the second basic problem of Combinatorics, namely

# 2 Partitioning $m$ objects into $k$ groups

This time, we need to specify whether

- the objects are distinct or identical,
- the group order is important or not,
- the order of objects (when *distinct*) within each group matters or not,
- we allow some groups to stay empty, or insist on each group containing at least one element.

We start with the three (easier) cases of

## 2.1 Ordered groups

This time, we derive only formulas for the total number of such partitions (rather then trying to generate them individually).

1. When the objects are indistinguishable, it is just a question of how many of them go into each group (cell, block, bin, box, shelf etc. - different names can be used here; same goes for 'objects', elements, letters, symbols, items, people, etc.).

   This is equivalent to selecting $m$ letters from the alphabet of $k$ letters (each letter representing a *cell*) with repetition allowed (cells can always have arbitrary number of objects) but order irrelevant (since both $aab$ and $aba$ would result in putting two objects into Cell $a$ and one object into Cell $b$, i.e. in *identical* partitioning, etc.). Note that this may result in one or more empty cells.

   The answer is thus given by (3), with $n$ being the number of *groups* $(= k)$ and $r$ being the number of objects $(= m)$, getting

   $$\binom{m+k-1}{m} = \binom{m+k-1}{k-1} \tag{6}$$

   Should we *not* allowing empty cells, we can put one object into each cell first (there is just one way of doing that) and then use the previous formula for placing the remaining $m - k$ objects into $k$ cells, getting

   $$\binom{m-1}{m-k} = \binom{m-1}{k-1}$$

2. When the objects are distinct and their order (within each group) matters, we first choose how many objects to put in each group - which can be done using (6). Then we fill in the 'blanks', one by one, each time selecting a

specific object. Since there are $m!$ possible ways to do the latter, each resulting in a different result (partition), the final answer is

$$m! \cdot \binom{k+m-1}{m} = \frac{(k+m-1)!}{(k-1)!} \tag{7}$$

*Not* allowing empty cells similarly results in

$$m! \cdot \binom{m-1}{k-1} \tag{8}$$

3. When objects are distinct and their order within each group irrelevant, all we have to do is to choose, for each object, the group to place it in (having $k$ choices every time). So, the answer is provided by (4) whose $n$ is now the number of groups ($= k$) and $r$ is the number of objects ($= m$), getting

$$k^m$$

To make sure that each cell contains *at least one* object is now more complicated (for details, see the next section); the corresponding EXPONENTIAL GENERATING FUNCTION (with respect to $m$, for $k$ fixed) is

$$\big(\exp(x) - 1\big)^k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \exp\big((k-i)\cdot x\big) \tag{9}$$

where $\exp(x) - 1$ is the EGF of the $0, 1, 1, 1...$ sequence (indicating that a cell is non-empty). The coefficient of $x^m$, multiplied by $m!$, yields the final answer of

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^m \tag{10}$$

This follows from the following, well-known Taylor expansion

$$\exp(a \cdot x) = 1 + a \cdot x + a^2 \frac{x^2}{2} + a^3 \frac{x^3}{3!} + a^4 \frac{x^4}{4!} + ...$$

## 2.2 Interchangeable (unordered) groups

This means that any rearrangement of groups (as long as their individual 'contents' remain the same) is not seen as a new partition. It is thus not very meaningful (even though still possible) to allow empty groups now; we therefore impose an extra condition that each group must contain at least one object (to allow empty groups would simply require using the 'non-empty' formula with 1, 2, ... and $k$ groups, then adding the answers).

### 2.2.1 Distinct objects, their order (within each group) relevant

Since each such solution (partitioning) is repeated exactly $k!$ times in (8), all we need to do is to divide that expression by $k!$, getting

$$\frac{m!}{k!} \cdot \binom{m-1}{k-1}$$

#### 2.2.2 Distinct objects, their order irrelevant

Similarly, each such solution is found $k!$ times in (10), implying that the new answer is

$$\left\{ {m \atop k} \right\} \stackrel{\text{def}}{=} \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^m$$

This expression is called the STERLING NUMBER OF THE SECOND KIND. The *ordinary* generating function of these (with respect to $m$, with $k$ fixed) is

$$\prod_{i=1}^{k} \frac{x}{1 - i \cdot x}$$

(see the next section).

#### 2.2.3 Identical objects

Now we are partitioning $m$ identical objects into exactly $k$ non-empty and 'interchangeable' (meaning: their order is irrelevant) groups; this is clearly the same as writing the integer $m$ as a sum of exactly $k$ *positive* integers. The answer is given by the PARTITION FUNCTION, usually denoted $P(m, k)$, having the following (ordinary) GF (with respect to $m$, $k$ fixed)

$$\prod_{i=1}^{k} \frac{x}{1 - x^i} \tag{11}$$

(see the next section). There is no simple explicit formula for the coefficients of the corresponding Taylor expansion.

# 3 Generating Functions

These are of two distinct types.

## 3.1 Ordinary GFs

The most general case: suppose there are several (typically many) choices to select a specific 'arrangement' of 'objects' (what these actually are varies from case to case). Let there also be a mapping from the set of these arrangements into non-negative integers), and let $g_\ell$ be the *number* of arrangements which map into a specific integer $\ell$. The (ordinary) generating function of these $g_\ell$'s is defined as

$$G(x) = \sum_{\ell=0}^{\infty} g_\ell x^\ell \tag{12}$$

In most cases, it turns out to be a finite product of several terms, found by breaking the task of constructing an arrangement into several **stages**, each stage having its own set of 'sub-arrangements'. These must also be mapped into integers, to yield their individual contributions to $\ell$ (defined as the resulting sum; $\ell$ thus needs to be 'stage-additive'). Furthermore, each 'total' arrangement must be achieved by *exactly one* specific sequence of sub-arrangements, implying the usual MULTIPLICATION PRINCIPLE: the numbers of sub-arrangements (one for each stage) need to be multiplied to yield the total number of arrangements.

The resulting OGF is then a simple *product* of the individual OGFs at each stage.

**Example 5** *In how many ways can we select $r$ **different** letters out of a collection of $n$ distinct letters when the resulting order is irrelevant.*

*In this example, selecting an 'arrangement' is deciding, for each of the $n$ letters, whether to keep it or not; each of these 'stages' contributes $0$ or $1$ (respectively) to $\ell$, defined as the number of letters **kept**. The OGF of the value contributed to $\ell$ at each stage is thus $1+x$ (the coefficient of each term represents the number of ways to make that choice, the exponent is the resulting $\ell$ contribution), implying that*

$$G(x) = (1+x)^n$$

*The answer is the coefficient of $x^r$ in the expansion of this function, which yields (2) - note that $\ell$ of the derivation has become $r$ of the original question (a dummy variable of the last expression). Also note that the final OGF thus encapsulates, for a given $n$, all the answers (in terms of $r$).*

**Example 6** *Find the OGF (with respect to $r$, keeping $n$ fixed) for the number of ways one can select $r$ letters out of a collection of $n$ distinct letters, with unlimited duplication allowed and the resulting order irrelevant.*

*This time we have to decide, for each of our $n$ letters, how many times to include it; we can still do this in $n$ stages. The contribution to $\ell$ (number of letters selected) has the following **one-stage** OGF - there is just one way (the coefficient) of duplicating a letter exactly $i$ (the exponent of $x$) times, where $i$ must go from $0$ to $\infty$ to cover any potential value of $\ell$:*

$$1 + x + x^2 + x^3 + x^4 + ... = \frac{1}{1-x}$$

*implying the overall OGF of*

$$\frac{1}{(1-x)^n}$$

**Example 7** *Find the OGF (with respect to $m$, keeping $k$ fixed) for the number of partitions of $m$ distinct objects into $k$ non-empty groups (their order irrelevant). Note that we now let $m$ to be the dummy variable (the old $\ell$) of (12).*

*We assign each of the $m$ objects to a group, one by one, by starting to fill one group until we need another, then another, ..., until we have exactly $k$ of them; starting a new group is always seen as the beginning of a new **stage**. To cover every possible value of $m$, we proceed as if this $m$ was infinite; it is the* *task (and beauty) of the resulting OGF have the answer for any and each finite $m$.*

*In Stage 1, we put Object 1 into a group, then keep on adding Object 2, .... to the same group (adding **any** number of extra objects, including* 0*, is possible); each time we do it, we have only one choice of an object. The first-stage OGF is thus (coefficients represent the number of choices, powers specify the corresponding contribution to $m$):*

$$x + x^2 + x^3 + ... = \frac{x}{1-x}$$

*In the second stage, we place the next available object to a **new** group, then keep on adding the next few (potentially any number) objects to **either one** of the already existing two groups; this time we have **two** choices for where to place each of the **additional** objects, getting, for the corresponding Stage-2 OGF*

$$x + 2x^2 + 2^2x^3 + 2^3x^4 + ... = \frac{x}{1-2x}$$

*In Stage 3, we start yet another group, then keep on adding (any number of) extra objects to one of the **three** existing groups (thus having* 3 *choices for each extra object); the OGF of this stage is thus*

$$x + 3x^2 + 3^2x^3 + 3^3x^4 + ... = \frac{x}{1-3x}$$

*This is repeated until filling exactly $k$ groups in this manner. Note that we have thus covered **all** possibilities of creating $k$ '**unordered**' (if they were ordered, we would have $k$ choices, not one, for placing Object 1, etc. ) and **non-empty** groups, using **any** (total) number of objects.*

*The overall OGF is thus*

$$\prod_{i=1}^{k} \frac{x}{1 - i \cdot x}$$

*The answer to our original question is the coefficient of $x^m$ in the Taylor expansion of this function.*

**Example 8** *In how many ways can we write an integer $m \geq k$ as a sum of **exactly** $k$ positive integers (their order does not count).*

*This is **equivalent** to writing $m$ as a sum of **any** number of positive integers, the largest of which (there may be more than one such term) equal(s) to $k$. This is based on a duality demonstrated by the following example:*

$$9 = 5 + 2 + 2 = 3 + 3 + 1 + 1 + 1$$

*where* 9 *is written as a sum of **three** terms, and also as a ('dual', in the one-to-one sense) sum of terms having the largest value of* 3*. The equivalence of the two can be seen (just count the dots, either horizontally or vertically) from the following* YOUNG DIAGRAM

$$\begin{array}{lllll} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & & \\ \bullet & \bullet & & & \end{array}$$

*We construct the desired OGF using the second version of the problem (largest term(s) equal to $k$) by multiplying OGFs of the individual $k$ stages. In Stage 1 we select $i$ number of 1's (there is one way of doing it, contributing $i$, which becomes the exponent of $x$, to the value of $m$) where $i$ can go from 0 to $\infty$, in Stage 2 we similarly select any number of 2's ($i$ of them now contribute $2i$ to the total), etc. The corresponding Stage-$s$ contribution to the final OGF is thus*

$$1 + x^s + x^{2s} + x^{3s} + ... = \frac{1}{1 - x^s}$$

*But note that the last stage is exceptional: we* ***must*** *select at least one $k$; its contribution to OGF is thus*

$$x^k + x^{2k} + x^{3k} + ... = \frac{x^k}{1 - x^k}$$

*Multiplying these (altogether $k$) one-stage contributions yields the overall OGF of the original question, namely*

$$\frac{x^k}{1 - x^k} \prod_{s=1}^{k-1} \frac{1}{1 - x^s}$$

*which agrees with (11). The final answer requires Taylor-expanding this function and taking the coefficient of $x^m$.*

**Example 9** *When rolling 15 dice, one gets a certain number of singlets (a specific number of dots appearing only once), doublets (twice), triplets (three times), etc., all the way to the same number of dots appearing 15 times; one such possibility is getting 5 triplets; another is getting 2 singlets, 3 doublets, 2 triplets and 1 quadruplet, etc. How many such possibilities are there?*

*This of course is the same as writing 15 as a sum of numbers* ***not bigger*** *than 6; to get the corresponding OGF, we need to modify the previous formula accordingly (by removing the $x^k$ factor, since $k$ now may also appear* ***zero*** *number of times). Expanding*

$$\prod_{s=1}^{6} \frac{1}{1 - x^s}$$

*yields*

$$1+x+2x^2+3x^3+5x^4+7x^5+11x^6+14x^7+20x^8+26x^9+35x^{10}+44x^{11}+58x^{12}+71x^{13}+90x^{14}+110x^{15}+...$$

*The answer is thus 110 (the generating function provides the answer for each and any number of dice rolled).*

## 3.2 Exponential GFs

Given a set of $\ell$ *distinct* elements, while $f(\ell)$ returns the **number** of specific 'arrangements' of these elements (called the set's $f$-count), the EXPONENTIAL

GENERATING FUNCTION of the $f(0)$, $f(1)$, $f(2)$, .... sequence is defined as

$$F(x) \stackrel{\text{def}}{=} \sum_{\ell=0}^{\infty} f(\ell) \frac{x^\ell}{\ell!}$$

(similar to defining its OGF, but having an extra $\ell!$ in the denominator).

**Example 10** *When $f$ returns the number of permutations of a set of $\ell$ distinct elements (i.e. $\ell!$), we get*

$$F(x) = \sum_{\ell=0}^{\infty} \ell! \cdot \frac{x^\ell}{\ell!} = \sum_{\ell=0}^{\infty} x^\ell = \frac{1}{1-x} \tag{13}$$

### 3.2.1 Main theorem

Starting with $k$ such functions (let us call them $h_j$, $j = 1, 2, ...k$) of a set's size, we now create a more complicated (but consistent with our original definition) COMPOSITE EGF by

$$F(x) = \sum_{\ell=0}^{\infty} \left( \sum_{\text{part}} \prod_{j=1}^{k} \hat{h}_j \right) \frac{x^\ell}{\ell!} \tag{14}$$

where the second summation is over all possible PARTITIONS of a set of $\ell$ distinct elements into *exactly $k$ ordered* (between, not within) *disjoint* subsets of arbitrary (including 0) size (as long as these add up to $\ell$), and $\hat{h}_j$ is $h_j$, applied to the $j^{\text{th}}$ subset.

One can then prove that another (simpler) way of constructing the same EGF is by

$$F(x) = \prod_{j=1}^{k} \left( \sum_{i=0}^{\infty} h_j(i) \frac{x^i}{i!} \right)$$

which is a product of $k$ *simple* EGFs, each built using just one of the $h_j$ functions.

In reverse, this enables us to *interpret* a product of two or more EGFs as a single EGF of the (14) type.

### 3.2.2 Important special case

happens when

- all $h_j$ functions be the same (one can then drop their subscript and call them $h$),
- where $h$ is the *indicator* function of a specific condition imposed on the **size** of each of the $k$ subsets ($h$ equals to 1 when the condition is met and to 0 otherwise).

This implies that the

$$\sum_{\text{part}} \prod_{j=1}^{k} \hat{h}_j$$

part of (14) is the *number* of ordered partitions of the original set into exactly $k$ disjoint subsets, *all* of which must meet, individually, this condition. Based on the main theorem, the corresponding EGF is equal to

$$F(x) = \left( \sum_{i=0}^{\infty} h(i) \frac{x^i}{i!} \right)^k \stackrel{\text{def}}{=} H(x)^k \tag{15}$$

**Example 11** *What is the number of distinct $r$-letter words based on $n$-letter alphabet (repetition allowed, order matters).*

*This is equivalent (but not necessarily easier than the direct approach used earlier) to partitioning a set of the first $r$ positive integers into $n$ distinct subsets of any size (thus listing the positions of Letter 1, Letter 2,... Letter $n$ in the corresponding $r$-letter word - an empty subset implies not using that letter at all). The EGF allowing each subset to be of* ***any*** *size (including 0) is*

$$H(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ... = \exp(x) \tag{16}$$

*The EGF which then answers our question (covering all possible values of $r$, for a fixed $n$) is*

$$F(x) = \exp(x)^n = \exp(n \cdot x)$$

*Note that its coefficient of $x^r$ (when Taylor-expanded in $x$) multiplied by $r!$, agrees with (4).*

**Example 12** *Same question, but this time* ***not*** *allowing any duplication of letters.*

*The only thing which changes in the solution is $H(x)$, which must now restrict the size of each subset to at most one element, thus:*

$$H(x) = 1 + x$$

*The coefficient of $x^r$ in the expansion of*

$$F(x) = (1 + x)^n$$

*further multiplied by $r!$ yields (5).*

**Example 13** *Find the number of different partitions of a set with $m$ distinct elements into and* ordered *collection of* non-empty *subsets.*

*Since $h$ is equal to 0 for an empty set and to 1 otherwise, we now get*

$$H(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4!} + ... = \exp(x) - 1 \tag{17}$$

*and*

$$F(x) = \big(\exp(x) - 1\big)^k$$

*The answer is $m!$ times the coefficient of $x^m$ in the Taylor expansion of this function, leading to (10).*

### 3.2.3 Composition of two EGFs

Suppose we have an EGF (usually of the simple, indicator-function type, which *must* exclude *empty* subsets) $H(x)$ on a set of elements, and another EGF, say $G(x)$, whose 'distinct objects' are disjoint non-empty *subsets* of the original set. It can be shown that the EGF for the number of distinct ways to combine the two (whatever the specifics are) is the COMPOSITION of $G$ and $H$ (be careful with the order, this operation is non-commutative), namely

$$G\left(H(x)\right)$$

We thus get the EGF for the number of ways $\ell$ elements of the original set can be partitioned into $k$ non-empty disjoint *unordered* (order is removed by the $k!$ denominator of $G$) subsets (all meeting Condition $H$), further multiplied by $g(k)$ then summed over all $k$ from 0 to $\infty$.

**Example 14** *We are supposed to rank a specific number of distinct items (e.g. in terms of quality), but we are allowed to put these into 'equivalence classes' whenever we feel that two or more of them are of the same quality. The question is: in how many different ways can this be done when having $\ell$ items to rank.*

*Here, we need $H(x)$ of (17) to ensure that subsets (the equivalence classes) are non-empty; furthermore, $G(x)$ must 'count' all possible permutations of such subsets. The resulting EGF is thus the composition of (13) and (17), namely*

$$\frac{1}{1-\big(\exp(x)-1\big)}=\frac{1}{2-\exp(x)}$$

*Expanding this function yields*

$$1+x+\frac{3}{2}x^2+\frac{13}{6}x^3+\frac{25}{8}x^4+\frac{541}{120}x^5$$

*and the first few answers, namely* 1, 1, 3, 13, 75, 541, ... *when having to rank* 0, 1, 2, 3, 4, 5, ... *items, respectively.*

**Example 15** *Find the number of* ***unordered*** *partitions of a set of $m$ distinct objects into* ***any*** *number of non-empty subsets.*

*The same $H$ ensures non-empty subsets; $G$ is now also an 'indicator' EGF, accepting* ***any*** *number of such subsets in the manner of (16). The resulting EGF is thus*

$$\exp\big(\exp(x)-1\big)$$

*resulting in* 1, 1, 2, 5, 15, 52, 203, ... *when partitioning a set of* 0, 1, 2, 3, 4, 5, 6,... *objects (these are the so-called* BELL NUMBERS*).*