

1 Quaternions and three dimensional rotations

One of the most elegant ways of performing rotations in three dimensions is to utilize the ALGEBRA OF QUATERNIONS. Its elements have the form of

$$A \oplus \mathbf{a} \tag{1}$$

where A and \mathbf{a} are referred to as the SCALAR and VECTOR parts, respectively. When the scalar part equals zero, we simplify the notation to \mathbf{a} . This means that, from now on, all vectors are considered to be *quaternions* with a zero scalar part.

Addition (subtraction) of two quaternions is a trivial, element-wise operation. So is multiplication of a quaternion by a scalar.

Multiplying two quaternions is done by

$$(A \oplus \mathbf{a}) \circ (B \oplus \mathbf{b}) = AB - \mathbf{a} \bullet \mathbf{b} \oplus A\mathbf{b} + B\mathbf{a} + \mathbf{a} \times \mathbf{b} \tag{2}$$

where \bullet and \times stand for the usual scalar and vector product, respectively. One can easily show that this multiplication is associative (no need for parentheses when multiplying three or more quaternions), but it is obviously *not* commutative (one cannot reverse the factors' order).

Proof of associativity: Post-multiplying (2) by $C \oplus \mathbf{c}$ yields:

$$\begin{aligned} & ABC - C(\mathbf{a} \bullet \mathbf{b}) - A(\mathbf{b} \bullet \mathbf{c}) - B(\mathbf{c} \bullet \mathbf{a}) - (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \oplus \\ & AB \mathbf{c} + AC \mathbf{b} + BC \mathbf{a} + C(\mathbf{a} \times \mathbf{b}) + A(\mathbf{b} \times \mathbf{c}) + B(\mathbf{a} \times \mathbf{c}) \\ & - (\mathbf{a} \bullet \mathbf{b})\mathbf{c} + (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{b} \bullet \mathbf{c})\mathbf{a} \end{aligned} \tag{3}$$

based on the well known

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{b} \bullet \mathbf{c})\mathbf{a} \tag{4}$$

Similarly, post-multiplying $A \oplus \mathbf{a}$ by

$$BC - \mathbf{b} \bullet \mathbf{c} \oplus B \mathbf{c} + C \mathbf{b} + \mathbf{b} \times \mathbf{c} \tag{5}$$

yields

$$\begin{aligned} & ABC - C(\mathbf{a} \bullet \mathbf{b}) - A(\mathbf{b} \bullet \mathbf{c}) - B(\mathbf{c} \bullet \mathbf{a}) - (\mathbf{b} \times \mathbf{c}) \bullet \mathbf{a} \\ & \oplus AB \mathbf{c} + AC \mathbf{b} + BC \mathbf{a} + C(\mathbf{a} \times \mathbf{b}) + A(\mathbf{b} \times \mathbf{c}) + B(\mathbf{a} \times \mathbf{c}) \\ & - (\mathbf{a} \bullet \mathbf{b})\mathbf{c} + (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{b} \bullet \mathbf{c})\mathbf{a} \end{aligned} \tag{6}$$

Due to the cyclic property of triple product, this is identical to (3).

By utilizing the corresponding Taylor series, we can define *functions* of quaternions, for example

$$\exp_{\circ}(A \oplus \mathbf{a}) = \exp A \cdot (\cos a \oplus \sin a \hat{\mathbf{a}}) \tag{7}$$

where $a \equiv |\mathbf{a}|$ and $\hat{\mathbf{a}} \equiv \frac{\mathbf{a}}{a}$.

Proof: Because $A \circ \mathbf{a} = \mathbf{a} \circ A$, we get

$$\exp_{\circ}(A \oplus \mathbf{a}) = \exp(A) \cdot \exp_{\circ}(\mathbf{a}) \quad (8)$$

where the first factor is clearly the regular \exp function of a scalar argument, and the second one can be expanded as follows

$$1 + a \hat{\mathbf{a}} - \frac{a^2}{2!} \hat{\mathbf{a}}^2 + \frac{a^3}{3!} \hat{\mathbf{a}}^3 - \frac{a^4}{4!} \hat{\mathbf{a}}^4 + \frac{a^5}{5!} \hat{\mathbf{a}}^5 - \dots \quad (9)$$

since $\hat{\mathbf{a}} \circ \hat{\mathbf{a}} = -\hat{\mathbf{a}} \bullet \hat{\mathbf{a}} = -1$, $\hat{\mathbf{a}} \circ \hat{\mathbf{a}} \circ \hat{\mathbf{a}} = -\hat{\mathbf{a}}$, $\hat{\mathbf{a}} \circ \hat{\mathbf{a}} \circ \hat{\mathbf{a}} \circ \hat{\mathbf{a}} = 1$, $\hat{\mathbf{a}} \circ \hat{\mathbf{a}} \circ \hat{\mathbf{a}} \circ \hat{\mathbf{a}} \circ \hat{\mathbf{a}} = \hat{\mathbf{a}}$, etc. Collecting, individually, the odd and even terms of (9) yields the second factor of (7).

A QUATERNIONIC CONJUGATE of $\mathbb{A} \equiv A \oplus \mathbf{a}$, denoted $\overline{\mathbb{A}}$, is defined as $A \oplus -\mathbf{a}$ (the vector part changes sign). It can be easily verified that

$$\overline{\mathbb{A} \circ \mathbb{B}} = \overline{\mathbb{B}} \circ \overline{\mathbb{A}} \quad (10)$$

Rotating (clock-wise) a vector \mathbf{r} around an axis passing through the origin and of direction $\hat{\mathbf{s}}$, by an angle equal to the magnitude of θ , is achieved by

$$\exp_{\circ}\left(\frac{\theta}{2}\hat{\mathbf{s}}\right) \circ \mathbf{r} \circ \overline{\exp_{\circ}\left(\frac{\theta}{2}\hat{\mathbf{s}}\right)} = \exp_{\circ}\left(\frac{\theta}{2}\hat{\mathbf{s}}\right) \circ \mathbf{r} \circ \exp_{\circ}\left(-\frac{\theta}{2}\hat{\mathbf{s}}\right) \quad (11)$$

Proof:

$$\begin{aligned} \exp_{\circ}\left(\frac{\theta}{2}\hat{\mathbf{s}}\right) \circ (\mathbf{r}_{\parallel} + \mathbf{r}_{\perp}) \circ \exp_{\circ}\left(-\frac{\theta}{2}\hat{\mathbf{s}}\right) &= \mathbf{r}_{\parallel} + \exp_{\circ}(\theta\hat{\mathbf{s}}) \circ \mathbf{r}_{\perp} = \\ \mathbf{r}_{\parallel} + (\cos \theta \oplus \sin \theta \hat{\mathbf{s}}) \circ \mathbf{r}_{\perp} &= \mathbf{r}_{\parallel} + \mathbf{r}_{\perp} \cos \theta + \sin \theta \hat{\mathbf{s}} \times \mathbf{r}_{\perp} \end{aligned} \quad (12)$$

where $\mathbf{r}_{\parallel} + \mathbf{r}_{\perp}$ is decomposing \mathbf{r} into parts parallel and perpendicular to $\hat{\mathbf{s}}$. Note that when two vectors (say \mathbf{q} and \mathbf{w}) are perpendicular, they anti-commute (when multiplied as two quaternions). This implies $\mathbf{q} \circ \exp_{\circ}(-\mathbf{w}) = \exp_{\circ}(\mathbf{w}) \circ \mathbf{q}$. Similarly, parallel vectors commute, implying $\mathbf{q} \circ \exp_{\circ}(-\mathbf{w}) = \exp_{\circ}(-\mathbf{w}) \circ \mathbf{q}$.

Note that the *same* rotation can be achieved by

$$-\exp_{\circ}\left(\frac{\theta}{2}\hat{\mathbf{s}}\right) = \exp_{\circ}\left(\frac{\theta-2\pi}{2}\hat{\mathbf{s}}\right) \quad (13)$$

which means that there is a 2 to 1 correspondence between quaternions of the $\exp_{\circ}\left(\frac{\theta}{2}\hat{\mathbf{s}}\right)$ type and rotations (i.e. quaternions have a different representation for rotating by, say 90° and -270° , around the same axis).

Let $\|\mathbb{A}\|$ denote a non-negative MAGNITUDE of $\mathbb{A} \equiv A \oplus \mathbf{a}$, defined by

$$\|\mathbb{A}\|^2 = \mathbb{A} \circ \overline{\mathbb{A}} = \overline{\mathbb{A}} \circ \mathbb{A} = A^2 + |\mathbf{a}|^2 \quad (14)$$

It is quite obvious that

$$\frac{\bar{\mathbb{A}}}{\|\mathbb{A}\|^2} \equiv \mathbb{A}^{-1} \quad (15)$$

provides an INVERSE of \mathbb{A} .

A quaternion, say $S \oplus \mathbf{s}$, whose magnitude is equal to 1 can always be written in the form of

$$\exp_{\circ}(\frac{\theta}{2}\hat{\mathbf{s}}) \quad (16)$$

(the reverse is quite obvious, in view of (7)).

Proof: Since $S^2 + |\mathbf{s}|^2 = 1$, it is possible to find $\theta \in [0, 2\pi)$, such that $S = \cos \frac{\theta}{2}$ and $|\mathbf{s}| = \sin \frac{\theta}{2}$. Using (7), we can now easily show that

$$\exp_{\circ}(\frac{\theta}{2}\hat{\mathbf{s}}) = \cos \frac{\theta}{2} \oplus \sin \frac{\theta}{2} \hat{\mathbf{s}} = S \oplus \mathbf{s} \quad (17)$$

When such an $\mathbb{S} \equiv S \oplus \mathbf{s}$ (which we can now interpret as a rotation) becomes a function of time t (rather than being a *fixed* quaternion), then

$$2\mathbb{S}' \circ \bar{\mathbb{S}} \quad (18)$$

(the prime implying element-wise differentiation with respect to t) is a *vector* whose direction agrees with the *instantaneous* axis of rotation, and whose length yields the corresponding angular speed.

Proof: Clearly,

$$(\mathbb{S} \circ \bar{\mathbb{S}})' = \mathbb{S}' \circ \bar{\mathbb{S}} + \mathbb{S} \circ \bar{\mathbb{S}}' = 0 \quad (19)$$

since $\mathbb{S} \circ \bar{\mathbb{S}} = 1$ and the product rule applies to quaternionic multiplication as well. This shows that $\mathbb{S}' \circ \bar{\mathbb{S}}$, added to its own conjugate, yields zero, implying that $\mathbb{S}' \circ \bar{\mathbb{S}}$ is a 'pure' vector.

Let's now rotate an arbitrary (fixed) vector \mathbf{r} to

$$\hat{\mathbf{r}} \equiv \mathbb{S} \circ \mathbf{r} \circ \bar{\mathbb{S}} \quad (20)$$

Differentiating $\hat{\mathbf{r}}$ with respect to t results in

$$\begin{aligned} \hat{\mathbf{r}}' &= \mathbb{S}' \circ \mathbf{r} \circ \bar{\mathbb{S}} + \mathbb{S} \circ \mathbf{r} \circ \bar{\mathbb{S}}' = \\ &= \mathbb{S}' \circ \bar{\mathbb{S}} \circ \hat{\mathbf{r}} + \hat{\mathbf{r}} \circ \mathbb{S} \circ \bar{\mathbb{S}}' = \\ &= (2\mathbb{S}' \circ \bar{\mathbb{S}}) \times \hat{\mathbf{r}} \end{aligned} \quad (21)$$

Based on a well know formula of classical mechanics, $2\mathbb{S}' \circ \bar{\mathbb{S}}$ is the corresponding angular velocity.

2 Geometry of a spherical surface

For simplicity, we assume that the sphere is of radius 1. To preserve the underlying symmetry, we will not introduce any coordinates on the surface itself (which, by necessity, would have to break this symmetry), but instead employ the usual Cartesian coordinates of the three-dimensional space in which the sphere is 'embedded' (placing the origin at the sphere's centre).

The individual POINTS of the surface are thus identified with three-dimensional vectors of unit length (seen here as a part of quaternion algebra of the previous section).

On the sphere's surface, a (two-dimensional) SPHERICAL VECTOR is defined as an *oriented segment* of a GREAT CIRCLE (a natural definition of a 'straight' path - great circles are those centered on the origin). Equivalently, any such vector can be represented by its starting point (its BUTT), say $\hat{\mathbf{a}}$, and a rotation (in the sense of the previous section) $\exp_{\circ}(\frac{\theta}{2}\hat{\mathbf{s}}) = \cos \frac{\theta}{2} \oplus \sin \frac{\theta}{2} \hat{\mathbf{s}}$, which moves $\hat{\mathbf{a}}$ along the corresponding great circle to the vector's TIP. Clearly, $\hat{\mathbf{s}}$ must be perpendicular (in the three-dimensional sense) to $\hat{\mathbf{a}}$, for this move to follow a great circle, and θ must equal to the segment's angular span (measured from the origin), which defines the vector's LENGTH. Its tip will thus have the following (three-dimensional) coordinates

$$\begin{aligned} & (\cos \frac{\theta}{2} \oplus \sin \frac{\theta}{2} \hat{\mathbf{s}}) \circ \hat{\mathbf{a}} \circ (\cos \frac{\theta}{2} \oplus -\sin \frac{\theta}{2} \hat{\mathbf{s}}) = & (22) \\ & (\cos \frac{\theta}{2} \oplus \sin \frac{\theta}{2} \hat{\mathbf{s}}) \circ (\cos \frac{\theta}{2} \oplus \sin \frac{\theta}{2} \hat{\mathbf{s}}) \circ \hat{\mathbf{a}} = \\ & (\cos \theta \oplus \sin \theta \hat{\mathbf{s}}) \circ \hat{\mathbf{a}} \end{aligned}$$

since $-\hat{\mathbf{a}} \circ \hat{\mathbf{s}} = \hat{\mathbf{s}} \circ \hat{\mathbf{a}}$ (the two vectors are orthogonal). Note that spherical vectors are *not* the 'pure' vectors of the previous section. From now on, all our vectors are spherical (and are simply called 'vectors').

Sliding a vector (a segment of a great circle) along this great circle yields what is called a 'free' vector (unattached to a specific starting point $\hat{\mathbf{a}}$). Such a vector is then fully specified by $\exp_{\circ}(\theta\hat{\mathbf{s}}) = \cos \theta \oplus \sin \theta \hat{\mathbf{s}}$ alone ($\hat{\mathbf{s}}$ is perpendicular to the vector's great circle, θ is the vector's length).

Two free vectors can be added by sliding them along their respective great circles, so that the butt of the second vector is joined to the tip of the first one. The two end points of the path thus created (from the butt of the first vector to the tip of the second one) can be connected, *directly*, by a third vector (thus completing the so called SPHERICAL TRIANGLE), whose quaternionic representation is

$$\begin{aligned} & (\cos \theta_2 \oplus \sin \theta_2 \hat{\mathbf{s}}_2) \circ (\cos \theta_1 \oplus \sin \theta_1 \hat{\mathbf{s}}_1) = & (23) \\ & \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \hat{\mathbf{s}}_1 \bullet \hat{\mathbf{s}}_2 \oplus \sin \theta_1 \cos \theta_2 \hat{\mathbf{s}}_1 + \cos \theta_1 \sin \theta_2 \hat{\mathbf{s}}_2 \\ & - \sin \theta_1 \sin \theta_2 \hat{\mathbf{s}}_1 \times \hat{\mathbf{s}}_2 \end{aligned}$$

One can then easily establish its length, angles at which it meets the other two sides, etc. One should note that, in our 'operator' notation, addition corre-

sponds to quaternionic multiplication in which the two vectors are ordered from *right to left*. Such an addition is clearly non-commutative. The corresponding commutator is given by (note that quaternionic inverse reverses vectors' orientation, enabling us to 'subtract'):

$$\begin{aligned}
& [\exp_{\circ}(\theta_1 \hat{\mathbf{s}}_1) \circ \exp_{\circ}(\theta_2 \hat{\mathbf{s}}_2)]^{-1} \circ \exp_{\circ}(\theta_2 \hat{\mathbf{s}}_2) \circ \exp_{\circ}(\theta_1 \hat{\mathbf{s}}_1) = & (24) \\
& 1 - 2 \sin^2 \theta_1 \sin^2 \theta_2 [1 - (\hat{\mathbf{s}}_1 \bullet \hat{\mathbf{s}}_2)^2] \oplus \\
& -2 \sin \theta_1 \cos \theta_2 (\sin \theta_2 \cos \theta_1 - \sin \theta_1 \cos \theta_2 \hat{\mathbf{s}}_1 \bullet \hat{\mathbf{s}}_2) \hat{\mathbf{s}}_1 \times \hat{\mathbf{s}}_2 \\
& +2 \sin \theta_1 \sin^2 \theta_2 \cos \theta_1 (\hat{\mathbf{s}}_1 - \hat{\mathbf{s}}_1 \bullet \hat{\mathbf{s}}_2 \hat{\mathbf{s}}_2) \\
& -2 \sin^2 \theta_1 \sin \theta_2 \cos \theta_2 (\hat{\mathbf{s}}_2 - \hat{\mathbf{s}}_1 \bullet \hat{\mathbf{s}}_2 \hat{\mathbf{s}}_1)
\end{aligned}$$

We should point out that a composition of two *rotations*, say $\exp_{\circ} \frac{\mathbf{w}_1}{2}$ followed by $\exp_{\circ} \frac{\mathbf{w}_2}{2}$, results in $\exp_{\circ} \frac{\mathbf{w}_3}{2}$, where $\frac{\mathbf{w}_3}{2}$ can now be seen to be the *vector sum* of $\frac{\mathbf{w}_1}{2}$ and $\frac{\mathbf{w}_2}{2}$.

When both θ_1 and θ_2 are proportional to an infinitesimal quantity ε , addition of two vectors results in the usual, *commutative* $1 \oplus \theta_1 \hat{\mathbf{s}}_1 + \theta_2 \hat{\mathbf{s}}_2 + \dots$, if quantities proportional to the second and higher powers of ε are discarded; this is how a TANGENT SPACE is defined, at any given point.

Adding two vectors is not the same as the so called PARALLEL TRANSPORT of one vector, say $\exp_{\bullet}(\theta_1 \hat{\mathbf{s}}_1)$ along another, 'rail' vector $\exp(\theta_2 \hat{\mathbf{s}}_2)$. The tip of the transported vector will be, in the end of the transport, at

$$\begin{aligned}
& \exp_{\circ}(\frac{\theta_2}{2} \hat{\mathbf{s}}_2) \circ \exp_{\circ}(\theta_1 \hat{\mathbf{s}}_1) \circ \exp_{\circ}(\frac{\theta_2}{2} \hat{\mathbf{s}}_2) \circ \hat{\mathbf{a}} = & (25) \\
& [\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \hat{\mathbf{s}}_1 \bullet \hat{\mathbf{s}}_2 \\
& \oplus \sin \theta_1 (\hat{\mathbf{s}}_1 - \hat{\mathbf{s}}_1 \bullet \hat{\mathbf{s}}_2 \hat{\mathbf{s}}_2 + \cos \theta_2 \hat{\mathbf{s}}_1 \bullet \hat{\mathbf{s}}_2 \hat{\mathbf{s}}_2) + \cos \theta_1 \sin \theta_2 \hat{\mathbf{s}}_2] \circ \hat{\mathbf{a}}
\end{aligned}$$

($\hat{\mathbf{a}}$ is the point where the two butts met in the beginning of the transport - $\hat{\mathbf{a}}$ is thus perpendicular to both $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$). This means that the transported vector is simply being rotated by $\exp(\frac{\theta_2}{2} \hat{\mathbf{s}}_2)$.

Note that when both θ_1 and θ_2 are proportional to an infinitesimal parameter ε , the last expression, expanded to the $O(\varepsilon^2)$ accuracy, yields:

$$\left(1 - \frac{\theta_1^2 + \theta_2^2}{2} - \theta_1 \theta_2 \hat{\mathbf{s}}_1 \bullet \hat{\mathbf{s}}_2 + \dots \oplus \theta_1 \hat{\mathbf{s}}_1 + \theta_2 \hat{\mathbf{s}}_2 + \dots \right) \circ \hat{\mathbf{a}} \quad (26)$$

This clearly shows that when the rôles of the two vectors are reversed (i.e. we transport the *second* one along the *first*), the tip of the transported vector reaches, to the $O(\varepsilon^2)$ accuracy, the same final point as it did in the original transport. If a discrepancy was found, it would be called TORSION (our space thus has a zero torsion).

Parallel transport can be carried out along any specific path. This is easy when the path is just a linked (tip to butt) collection of several vectors: all we need to do is the composition of the corresponding rotations, which can be converted into a *single rotation* (this is true for any parallel transport).

When the path is defined, parametrically, by $\mathbf{r}(t)$, we need to solve

$$\mathbb{S}'(t) = \frac{\mathbf{r}(t) \times \mathbf{r}'(t)}{2} \circ \mathbb{S}(t) \quad (27)$$

where $\mathbb{S}(0) = 1$, and t varies from 0 to T . This amounts to solving four linear differential equations for the individual components of $\mathbb{S}(t)$. To parallel-transport a vector, say $\exp_{\circ}(\theta \hat{\mathbf{s}})$, along this path, we rotate its tip by

$$\mathbb{S}(T) \circ \exp_{\circ}(\theta \hat{\mathbf{s}}) \circ \overline{\mathbb{S}(T)} \quad (28)$$

while its butt ends up at $\mathbf{r}(T)$.

We will not give any examples of solving (27), which can usually be done only numerically, but we will show that $\|\mathbb{S}(t)\| = 1$, which makes $\mathbb{S}(t)$ a rotation.

Proof:

$$\begin{aligned} (\mathbb{S} \circ \overline{\mathbb{S}})' &= \mathbb{S}' \circ \overline{\mathbb{S}} + \mathbb{S} \circ \overline{\mathbb{S}'} \\ &= \frac{\mathbf{r} \times \mathbf{r}'}{2} \circ \mathbb{S} \circ \overline{\mathbb{S}} - \mathbb{S} \circ \overline{\mathbb{S}} \circ \frac{\mathbf{r} \times \mathbf{r}'}{2} = 0 \end{aligned} \quad (29)$$

since

$$\overline{\frac{\mathbf{r} \times \mathbf{r}'}{2}} = -\frac{\mathbf{r} \times \mathbf{r}'}{2} \quad (30)$$

and $\mathbb{S} \circ \overline{\mathbb{S}} = 1$ initially. This implies that $\|\mathbb{S}\|$ will have the same, constant value (equal to 1) for any t .

Finally, we combine vector addition and parallel transport in the following manner: an arbitrary vector is transported, first along $\exp_{\circ}(\theta_1 \hat{\mathbf{s}}_1)$ and then along $\exp_{\circ}(\theta_2 \hat{\mathbf{s}}_2)$, where both θ_1 and θ_2 are $O(\varepsilon)$ small. After that, the same vector is transported (from its *initial* location) with the *order* of the *rail* vectors *reversed*. The corresponding difference (in the end location of the tip of the transported vector) defines the so called space's CURVATURE.

What we need to do first (to compute this difference) is to replace, in (23), θ_1 by $\frac{\theta_1}{2}$ and θ_2 by $\frac{\theta_2}{2}$ (which converts them into the desired rotations) and expand the answer to the $O(\varepsilon^2)$ accuracy. This yields

$$\mathbb{S} \equiv 1 - \frac{\theta_1^2 + \theta_2^2}{8} - \frac{\theta_1 \theta_2}{4} \hat{\mathbf{s}}_1 \bullet \hat{\mathbf{s}}_2 \oplus \frac{\theta_1}{2} \hat{\mathbf{s}}_1 + \frac{\theta_2}{2} \hat{\mathbf{s}}_2 - \frac{\theta_1 \theta_2}{4} \hat{\mathbf{s}}_1 \times \hat{\mathbf{s}}_2 + \dots \quad (31)$$

An arbitrary vector $\cos \theta_0 \oplus \sin \theta_0 \hat{\mathbf{s}}_0$ is then correspondingly transported to

$$\begin{aligned} \mathbb{S} \circ (\cos \theta_0 \oplus \sin \theta_0 \hat{\mathbf{s}}_0) \circ \overline{\mathbb{S}} &= \cos \theta_0 \oplus \\ \sin \theta_0 &\left[\left(1 - \frac{\theta_1^2 + \theta_2^2}{2} - \theta_1 \theta_2 \hat{\mathbf{s}}_1 \bullet \hat{\mathbf{s}}_2 \right) \hat{\mathbf{s}}_0 + \theta_1 \hat{\mathbf{s}}_1 \times \hat{\mathbf{s}}_0 + \theta_2 \hat{\mathbf{s}}_2 \times \hat{\mathbf{s}}_0 \right. \\ &\left. + \frac{\theta_1^2}{2} \hat{\mathbf{s}}_1 \bullet \hat{\mathbf{s}}_0 \hat{\mathbf{s}}_1 + \frac{\theta_2^2}{2} \hat{\mathbf{s}}_2 \bullet \hat{\mathbf{s}}_0 \hat{\mathbf{s}}_2 + \theta_1 \theta_2 \hat{\mathbf{s}}_2 \bullet \hat{\mathbf{s}}_0 \hat{\mathbf{s}}_1 \right] \end{aligned} \quad (32)$$

Interchanging the order of $\exp_{\circ}(\frac{\theta_1}{2}\hat{\mathbf{s}}_1)$ and $\exp_{\circ}(\frac{\theta_2}{2}\hat{\mathbf{s}}_2)$ yields an identical result, except for the last term. The difference between the two transports thus yields (taking θ_0 to be $O(\varepsilon)$ -small as well)

$$\theta_0\theta_1\theta_2(\hat{\mathbf{s}}_2 \bullet \hat{\mathbf{s}}_0 \hat{\mathbf{s}}_1 - \hat{\mathbf{s}}_1 \bullet \hat{\mathbf{s}}_0 \hat{\mathbf{s}}_2) = \theta_0\theta_1\theta_2(\hat{\mathbf{s}}_2 \times \hat{\mathbf{s}}_1) \times \hat{\mathbf{s}}_0 \quad (33)$$

to the $O(\varepsilon^3)$ accuracy.

The result is obviously antisymmetric with respect to the $1 \leftrightarrow 2$ interchange; it also meets the BIANCHI IDENTITY (cyclically rotating the *three* indices and adding the resulting expressions yields zero). This can be seen from

$$\hat{\mathbf{s}}_2 \bullet \hat{\mathbf{s}}_0 \hat{\mathbf{s}}_1 - \hat{\mathbf{s}}_1 \bullet \hat{\mathbf{s}}_0 \hat{\mathbf{s}}_2 + \hat{\mathbf{s}}_0 \bullet \hat{\mathbf{s}}_1 \hat{\mathbf{s}}_2 - \hat{\mathbf{s}}_2 \bullet \hat{\mathbf{s}}_1 \hat{\mathbf{s}}_0 + \hat{\mathbf{s}}_1 \bullet \hat{\mathbf{s}}_2 \hat{\mathbf{s}}_0 - \hat{\mathbf{s}}_0 \bullet \hat{\mathbf{s}}_2 \hat{\mathbf{s}}_1 = 0 \quad (34)$$

Exploring rotations of a spherical surface has thus given us not only better understanding of its rather nontrivial geometry, but also an introduction to basic concepts of curved space - a stepping stone to equations of General Relativity.

References

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