BROCK UNIVERSITY

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Name	**Student Number**

[20] 1. There are 10 multiple choice questions and they are worth 2 points each. Circle the right answer for each of the following:

(1) Which of the following sets is <u>not</u> a subspace of M_{22}.

(A) All 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + c + 1 = d$.

(B) All 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $c + d = 0$.

(C) All 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + c = d$.

(D) All 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + b + c = 0$.

(E) All 2×2 matrices A such that $A = A^t$.

(2) What is the dimension of the subspace of M_{22} consisting of all 2×2 matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with $a + d = 0$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

(3) For which value of k, the vectors $v_1 = x - 2x^2, v_2 = 1 + x + x^2,$ and $v_3 = 1 + 2x - kx^2$ are linearly dependent in P_2.

(A) 2 **(B)** 1 **(C)** 0 **(D)** -1 **(E)** None of the others.

(4) If an 2×2 matrix $M = P^{-1} \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} P$, where $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, then M^5 is

(A) $\begin{bmatrix} 243 & 484 \\ -242 & -485 \end{bmatrix}$

(B) $\begin{bmatrix} 241 & 484 \\ -242 & -485 \end{bmatrix}$

(C) $\begin{bmatrix} 241 & 486 \\ -242 & -486 \end{bmatrix}$

(D) $\begin{bmatrix} 241 & 484 \\ -242 & -486 \end{bmatrix}$

(E) None of the others.

(5) Let $T : R^6 -> R^5$ be a linear transformation. If the nullity of T is 3, then the rank of T is

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

(6) Let $B = \{f_1, f_2, f_3\}$ be the basis for P_2, where $f_1 = 1 + x, f_2 = 1 - 2x, f_3 = 1 + x + x^2$. Then the coordinate matrix of $2 + 5x + 2x^2$ relative to B is

(A) $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ **(B)** $\begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}$ **(C)** $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ **(D)** $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ **(E)** None of the others.

(7) Let $T : R^3 \longrightarrow R^2$ be the linear transformation such that $T(1, 0, 0) = (1, 1), T(1, 0, 1) = (0, 4), T(1, 1, 1) = (1, 2)$. Then $T(2, 2, 1)$ is

(A) $(3, 1)$ **(B)** $(-3, 1)$ **(C)** $(3, -1)$ **(D)** $(-1, -3)$ **(E)** $(-3, -1)$

(8) Let $T : P_2 \longrightarrow P_1$ be the linear transformation defined by $T(a_0 + a_1x + a_2x^2) = (a_0 + 3a_1) - (2a_0 - a_1 + 3a_2)x$. Then the matrix $[T]_{B',B}$ for T with respect to bases $B = \{1, x, x^2\}$ and $B' = \{1, x\}$ for P_2 and P_1 is

(A) $\begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & -3 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 3 & 0 \\ -2 & -1 & -3 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 3 & 0 \\ -2 & 1 & 3 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 3 & 0 \\ -2 & 1 & -3 \end{bmatrix}$

(E) None of the others.

(9) Which of the following is equal to the complex number $\frac{2i}{(1+i)(1+3i)(1-3i)}$.

(A) $\frac{1+i}{5}$ **(B)** $\frac{1-i}{10}$ **(C)** $\frac{-1-i}{10}$ **(D)** $\frac{-1+i}{10}$ **(E)** $\frac{1+i}{10}$

(10) Which of the following is equal to $C(BA)$, where $A = \begin{bmatrix} 3+2i & 0 \\ 0 & 2 \\ -1+i & 1-i \end{bmatrix}$,

$B = \begin{bmatrix} -1-i & 0 & -i \\ 0 & 2i & -5 \end{bmatrix}$, $C = \begin{bmatrix} i & -1 \\ 0 & i \end{bmatrix}$.

(A) $\begin{bmatrix} 1+5i & 6-10i \\ 5-5i & -9-5i \end{bmatrix}$

(B) $\begin{bmatrix} -1+5i & 6-10i \\ 5+5i & 1-5i \end{bmatrix}$

(C) $\begin{bmatrix} -1+5i & 6+10i \\ 5+5i & 1-5i \end{bmatrix}$

(D) $\begin{bmatrix} -1+5i & 6-10i \\ 5+5i & 9-5i \end{bmatrix}$

(E) None of the others.

[7] 2. Consider the basis $B = \{v_1, v_2, v_3\}$ for R^3 (with the Euclidean inner product), where $v_1 = (1, 1, 0), v_2 = (3, 1, 0), v_3 = (0, 1, 4)$.

(a) Is B an orthonormal basis? (explain briefly.)

(b) If the answer to (a) is “no”, convert B to an orthonormal basis B' by applying the Gram -Schmidt process.

(c) Let $u = (0, 1, 1)$. Find $(u)_{B'}$ the coordinate vector of u relative to the orthonormal basis B'.

[10] 3. Given the weighted Euclidean inner product defined by $< u, v >= 3u_1\bar{v}_1 + 5u_2\bar{v}_2$ in C^2, and let $u = (-i, 1), v = (i, 3i)$.

(a) Find $< u, v >, ||u||$ and $d(u, v)$.

(b) Verify $< u, v >= \overline{< v, u >}$.

(c) Let $w = tu + v$. Find the value of t so that w and v are orthogonal.

(d) Define a new function $< u, v >= u_1\bar{v}_1 - 3u_2\bar{v}_2(*)$ for any two vectors $u = (u_1, u_2), v = (v_1, v_2)$ in C^2. Is this function an inner product in C^2? (why or why not?)

[10] 4. Let $T : R^4 \longrightarrow R^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ 2x_1 + 5x_2 + x_3 + x_4 \\ 3x_1 + 4x_2 - 9x_3 + 5x_4 \end{bmatrix}.$$

(a) Find the standard matrix A for T.

(b) Find a basis for the kernel of T and nullity(T).

(c) Find a basis for the range of T and rank(T).

(d) State the dimension theorem for linear transformations and verify it with the above T.

(e) Is the vector $\begin{bmatrix} 5 \\ 9 \\ 10 \end{bmatrix}$ in the range of T.

(f) Is the vector $\begin{bmatrix} 4 \\ -2 \\ 1 \\ 1 \end{bmatrix}$ in the kernel of T.

[8] 5. Let $T : R^2 \longrightarrow R^2$ be the linear transformation defined by $T(x_1, x_2) = (3x_1 + x_2, x_1 - x_2)$ and $B = \{u_1, u_2\}$ and $B' = \{v_1, v_2\}$ be bases for R^2, where $u_1 = (1, 1), u_2 = (1, -1)$ and $v_1 = (1, 1), v_2 = (1, 5)$.

(a) Find the transition matrix P from B' to B.

(b) Find P^{-1}.

(c) Given $[X]_B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Find $[X]_{B'}$.

(d) Find $[T]_{B',B}$ the matrix for T with respect to the bases B, B'.

[10] 6. Let $T : P_2 \longrightarrow P_2$ be the linear operator such that $T(a_0 + a_1x + a_2x^2) = 3a_0 + (a_0 + 2a_2)x + (-a_0 + 3a_1 + a_2)x^2$.

(a) Find the matrix A for T with respect to the standard basis $B = \{1, x, x^2\}$.

(b) Find the eigenvalues of T.

(c) Find bases for the eigenspaces of T.

(d) Find an invertible matrix P which diagonalizes matrix A (i.e. $P^{-1}AP$ is diagonal).

(e) Find a basis B' for P_2 such that $[T]_{B'}$ the matrix for T with respect to the basis B' is diagonal and write out this diagonal matrix.

[15] 7. Do only three (3) of the following:

(a) Let V be a real inner product space and u, v be any vectors in V. Show that if u and v are orthogonal, then $||2u - 3v||^2 = 4||u||^2 + 9||v||^2$.

(b) Prove that if matrices A and B are similar, then A^3 and B^3 have the same eigenvalues (**Hint**: Prove that $|\lambda I - B^3| = |\lambda I - A^3|$.)

(c) Suppose that λ_1, λ_2 are distinct eigenvalues of the matrix A, and X_1, X_2 are eigenvectors corresponding to λ_1, λ_2 respectively. Show that X_1, X_2 are linearly independent.

(d) Show that if A has an eigenvalue λ, then $2A + A^2$ has an eigenvalue $2\lambda + \lambda^2$.

(e) Let $T : V \longrightarrow W$ be a linear transformation. Show that $Ker(T)$ the kernel of T is a subspace of V.

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