BROCK UNIVERSITY

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Course: Math 2P13	**Number of students: 13**
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Name	**Student Number**

Marks

[10] 1. Answer True or False to the following statements.

(a) Subsets of linearly dependent sets are linearly dependent. (   )

(b) If S is a linearly independent set, then non of vectors in S is a linear combination of other vectors. (   )

(c) Let $T, U : V -> V$ be linear and β be an ordered basis for V. If $[T]_\beta = [U]_\beta$, then $T = U$. (   )

(d) Let $T : V -> W$ be linear and β, γ be ordered bases for V and W respectively. If $\dim(V) = m, \dim(W) = n$, then $[T]_\beta^\gamma$ is an $n \times m$ matrix. (   )

(e) An elementary matrix is not invertible. (   )

(f) Let $T : V -> V$ be linear and β be an ordered basis for V. If $[T]_\beta$ is a diagonal matrix, then β may not be a basis consisting of eigenvectors of T. ( )

(g) The Eigenspace E_λ is not T -invariant. ( )

(h) Let $A = \begin{bmatrix} 2 & +i \\ -i & 3 \end{bmatrix}$. Then A is self -adjoint. ( )

(i) If V is an inner product space and $x, y, z \in V$ such that $< x, y - z >= 0$, then $y = z$.
. ( )

(j) If T is unitary and λ is an eigenvalue of T, then $|\lambda| = 1$. ( )

[10] 2. There are 5 multiple choice questions and they are worth 2 points each. Circle the right answer for each of the following:

(1) In C^2, define an inner product by $< x, y >= xAy^*$, where $A = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$. Which of the following is $< x, y >$ for $x = (1 - i, 2 + 3i)$ and $y = (2 + i, 3 - 2i)$.

(A) $6 + 21i$ **(B)** $6 - 21i$ **(C)** $-6 + 21i$ **(D)** $-6 - 21i$ **(E)** None of the others.

(2) Let $T : P_2(R) -> R^3$ be linear transformation defined by $T(a + bx + cx^2) = (a - 3b, b + c, a + b + 2c)$ and let β and γ be the standard bases for $P_2(R)$ and R^3 respectively. which of the following matrix is equal to $[T]_\beta^\gamma$.

(A) $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & -3 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(E) None of the others.

(3) Let $S = \{(i,0,1),(1,1,1)\}$ in C^3. Which of the following is true.(where $S^{\perp}$ is the orthogonal complement of S).

(A) $dim(S^{\perp}) = 3$ **(B)** $dim(S^{\perp}) = 2$ **(C)** $dim(S^{\perp}) = 1$ **(D)** $dim(S^{\perp}) = 0$ **(E)** None of the others.

(4) Let λ be an eigenvalue of T with multiplicity 4. Which of the following statement is not true.

(A) $dim(E_{\lambda})$ might be 1. **(B)** $dim(E_{\lambda})$ might be 2. **(C)**$dim(E_{\lambda})$ might be 3.

(D)$dim(E_{\lambda})$ might 4. **(E)** None of the others.

(5) Let $T : R^4 -> R^4)$ be linear transformations defined by $T(a,b,c,d) = (a+b, b-c, a+c, a+d)$ and $e_2 = (0,1,0,0)$. Then a basis for T-cyclic subspace generated by e_2 is

(A) $\{e_2\}$. **(B)** $\{e_2, T(e_2)\}$. **(C)** $\{e_2, T(e_2), T^2(e_2)\}$. **(D)** $\{e_2, T(e_2), T^2(e_2), T^3(e_2)\}$.

(E) None of the others.

[12] 3. (a) Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -5 & -1 \\ 8 & -3 & 2 \end{bmatrix}$, and $B = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -5 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. Find elementary matrices

E_1, E_2, and E_3 such that $E_3E_2E_1A = B$.

(b) Find the value of k that satisfies the following equation.

$$\det\begin{bmatrix} 5a_1 & 5a_2 & 5a_3 \\ 4b_1+5c_1 & 4b_2+5c_2 & 4b_3+5c_3 \\ 7a_1+3c_1 & 7a_2+3c_2 & 7a_3+3c_3 \end{bmatrix} = k\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

[9] 4. Let T be the linear operator on $P_2(R)$ defined by $T(f(x)) = f(x) + (x+1)f'(x)$.

(a) Compute $T(x)$ and $T^2(x)$.

(b) Find a basis for the T-cyclic subspace generated by x.

(c) Find matrices $[T]_\beta, [T^*]_\beta$, where β is the standard basis for $P_2(R)$, and T^* is the adjoint operator of T.

(d) Find $[T^*(x)]_\beta$ and $T^*(x)$.

[12] 5. Let $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$.

(a) Find all eigenvalues of A.

(b) Find a basis for each eigenspace of A.

(c) Find an orthonormal basis for R^3 consisting of eigenvectors of A.

(d) Find an orthogonal matrix P and diagonal matrix D such that $P^T AP = D$.

[12] 6. Let T be a linear operator on a finite dimensional vector space V with Jordan canonical form

$$\begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

(a) Find the characteristic polynomial, eigenvalues and their multiplicities of T.

(b) Find dot diagram corresponding to each eigenvalue λ_i of T.

(c) Find $\dim(K_{\lambda_i})$ and $dim(E_{\lambda_i})$ for each eigenvalue λ_i. For which $\lambda_i, K_{\lambda_i} = E_{\lambda_i}$? (Hint use the Jordan form of T)

(d) According to the dot diagram found in (b), how many cycles are there corresponding to the eigenvalue $\lambda = 5$? what are their lengths?

(e) What is the smallest positive integer p for which $K_5 = N((T - 5I)^p)$.

[15] 7. Do only three (3) of the following:

(a) Let T be a linear operator. Then v is an eigenvector of T corresponding to λ if and only if $v \neq 0, v \in N(T - \lambda I)$.

(b) For any square matrix A, prove that A and A^T have the same characteristic polynomial.

(c) Let T and U be self-adjoint operators on an inner product space V. Prove that T commutes with U if and only if TU is self-adjoint.

(d) Let β be a basis for an inner product space V, and let $y, z \in V$. Prove that $y = z$ if and only if $< y, v >=< z, v >$ for every $v \in \beta$.

(e) Given that $\det \begin{bmatrix} A & O \\ B & I \end{bmatrix} = \det(A)$ and $\det \begin{bmatrix} I & O \\ D & C \end{bmatrix} = \det(C)$, where A, C are square matrices, I is an identity matrix, and O is a zero matrix. Prove that $\det \begin{bmatrix} A & O \\ B & C \end{bmatrix} = \det(A)\det(C)$.

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[80]