SOME RESULTS ON HYPERCENTRAL UNITS IN INTEGRAL GROUP RINGS

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ABSTRACT. In this note we investigate the hypercentral units in integral group rings $\mathbb{Z}G$, where G is not necessarily torsion. One of the main results obtained is the following (Theorem 3.5): if the set of torsion elements of G is a subgroup T of G and if $Z_2(\mathcal{U})$ is not contained in $C_{\mathcal{U}}(T)$, then T is either an Abelian group of exponent 4 or a Q^* group. This extends our earlier result on torsion group rings.

1 Introduction

Let $\mathbb{Z}G$ denote the integral group ring of a group G and $\mathcal{U}(\mathbb{Z}G)$ the group of units of such a group ring. For convenience, we will sometimes write $\mathcal{U}$ instead of $\mathcal{U}(\mathbb{Z}G)$. If G is any group, let $Z_n(G)$ denote the n'th term of the ascending central series of G and let $\tilde{Z}(G) = \cup Z_n(G)$, the hypercentre of G.

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When G is finite, Arora, Hales and Passi [1] showed that $\tilde{Z}(\mathcal{U}) = Z_2(\mathcal{U})$ while Arora and Passi [2] showed that $\tilde{Z}(\mathcal{U}) \subseteq G \cdot Z(\mathcal{U})$ (and also completely determined $G \cap \tilde{Z}(\mathcal{U})$). These results were extended to torsion groups in [5] and [7].

Clearly, in general the equality, $\tilde{Z}(\mathcal{U}) = Z_2(\mathcal{U})$, will no longer hold when G is not necessarily torsion (consider torsion free nilpotent G). Our goal in this paper is to investigate the inequality $\tilde{Z}(\mathcal{U}) \subseteq G \cdot Z(\mathcal{U})$ in a more general setting. The obvious generalization of the torsion result would be again to prove that $\tilde{Z}(\mathcal{U}) \subseteq G \cdot Z(\mathcal{U})$. A weaker generalization of the torsion result would be to prove that $\tilde{Z}(\mathcal{U}) \subseteq G \cdot C_{\mathcal{U}}(T)$.

In the next section we prove (Theorem 2.3) that the second property holds whenever the torsion elements of G form an Abelian subgroup T and $\mathbb{Z}(G/T)$ has only trivial units. In the very particular case that G/T is cyclic and every finite subgroup of G is normal in G, we further obtain (Theorem 2.7) a complete description of $Z_n(\mathcal{U})$ for all $n \geq 1$ (in terms of $Z(\mathcal{U})$). In addition, we show that the stronger first property holds for various classes of groups.

The most substantial results in the paper are contained in section 3 and are concerned with the second centre $Z_2(\mathcal{U})$. Our main result there is Theorem 3.5, which characterizes T whenever $Z_2(\mathcal{U})$ is not contained in $C_{\mathcal{U}}(T)$.

2 Hypercentre

We begin with two lemmas which will be required later. The proof of the first follows familiar lines but will be included for completeness.

Lemma 2.1

If u is a hypercentral unit of finite order in $\mathbb{Z}G$ then u is trivial.

Proof

Say $u \in Z_n(\mathcal{U})$ is of finite order. Since $u^* \in Z_n(\mathcal{U})$ is also of finite order, and the torsion elements of the nilpotent group $Z_n(\mathcal{U})$ form a subgroup, uu^* must be of finite order. But uu^* has nonzero identity coefficient, so $uu^* = 1$ ([9], p.277). Hence $u \in \pm G$.

□

Lemma 2.2

Assume the torsion elements of G form a subgroup T. Then $\tilde{Z}(\mathcal{U}) \subseteq N_{\mathcal{U}}(T)$.

Proof

We will prove by induction on n that $Z_n(\mathcal{U}) \subseteq N_{\mathcal{U}}(T)$ for all $n \geq 1$. The case $n = 1$ is obvious, so assume that the result holds for $n = k, k \geq 1$.

Let $u \in Z_{k+1}(\mathcal{U})$. If $t \in T$, then $utu^{-1} = t\alpha$ for some $\alpha \in Z_k(\mathcal{U})$. By the induction hypothesis, $\alpha \in N_{\mathcal{U}}(T)$. Since $t^m = 1$ for some $m, 1 = ut^m u^{-1} = (utu^{-1})^m = (t\alpha)^m = t_1\alpha^m$ for some $t_1 \in T$. Thus $\alpha^m \in T$, and so α is a hypercentral unit of finite order. Lemma 2.1 says $\alpha \in \pm T$ and so $u \in N_{\mathcal{U}}(T)$ as desired.

□

The weaker property mentioned in the introduction now follows easily.

Theorem 2.3

Assume the torsion elements of G form an Abelian subgroup T and $\mathbb{Z}(G/T)$ has only trivial units. Then $\tilde{Z}(\mathcal{U}) \subseteq G \cdot C_{\mathcal{U}}(T)$.

Proof

Let $u \in \tilde{Z}(\mathcal{U})$. Lemma 2.2 tells us that $u \in N_{\mathcal{U}}(T)$, and thus $ug \in N_{\mathcal{U}}(T)$ for every $g \in G$. Since $\mathbb{Z}(G/T)$ has only trivial units, we know that $ug \in 1 + (\mathbb{Z}G)\Delta T$ $(*)$ for some $g \in G$. We will complete the proof by showing $ug \in C_{\mathcal{U}}(T)$.

Let $t \in T$ and ug as above. Then $(ug)t(ug)^{-1} = t'$, i.e. $(ug)t = t'(ug)$ for some $t' \in T$. This together with $(*)$ gives that $t + \alpha t = t' + t'\alpha$ for some $\alpha \in (\mathbb{Z}G)\Delta T$. Focusing on elements of T, we get that $(t - t')(1 + \beta) = 0$ for some $\beta \in \Delta(T)$, i.e. $(1 - t^{-1}t')(1 + \beta) = 0$. But this is impossible unless $t = t'$ and we are done.

□

Our investigation of the stronger property $\tilde{Z}(\mathcal{U}) \subseteq G \cdot Z(\mathcal{U})$ proceeds via an intermediate step. We first show that $\tilde{Z}(\mathcal{U}) \subseteq N_{\mathcal{U}}(G)$ for a certain class of groups, and then note that the normalizer property $N_{\mathcal{U}}(G) = G \cdot Z(\mathcal{U})$

holds for these groups. Until recently it was an open problem as to whether this latter property holds in general. While a counterexample has recently been found [3], the property does hold for many groups and we use that here (see [9] for more detail on the normalizer problem).

Lemma 2.4

Let G be a group with the property that whenever $x, y \in G$, there exists an integer n (depending on x and y) such that x^n and y commute. Then $\tilde{Z}(\mathcal{U}) \subseteq N_{\mathcal{U}}(G)$

Proof.

Using an argument similar to that seen in the proof of Lemma 2.2, we will prove by induction that $Z_n(\mathcal{U}) \subseteq N_{\mathcal{U}}(G)$ for all $n \geq 1$. If $n = 1$, the result is obvious. So assume the result holds for $n = k, k \geq 1$, and let $u \in Z_{k+1}(\mathcal{U})$.

If $g \in G$, then $ugu^{-1} = g\alpha$ for some $\alpha \in Z_k(\mathcal{U})$. By the induction hypothesis, $\alpha \in N_{\mathcal{U}}(G)$. Since some power g^m of g commutes with u, we have

$$g^m = ug^m u^{-1} = (g\alpha)^m = g_1 \alpha^m$$

for some $g_1 \in G$.

So $\alpha^m \in G$, and thus $\alpha^m(\alpha^m)^* = 1$. But $\alpha \in N_{\mathcal{U}}(G)$ implies $\alpha^*\alpha$ is central ([9], p.31), and so $\alpha^*\alpha = \alpha\alpha^*$. Thus we have $(\alpha\alpha^*)^m = 1$ and so $\alpha\alpha^*$ is a central unit of finite order. Hence $\alpha\alpha^* \in G$, so $\alpha \in G$ and $u \in N_{\mathcal{U}}(G)$ as desired.

□

FC groups satisfy the conditions of Lemma 2.4. In the next result, parts (i) and (ii) could be stated more generally but we have opted for conciseness.

Proposition 2.5

The inequality $\tilde{Z}(\mathcal{U}) \subseteq G \cdot Z(\mathcal{U})$ holds in each of the following cases:

(i) G is FC and has no 2-torsion.

(ii) G is FC and is locally nilpotent.

(iii) The torsion elements of G form an Abelian subgroup T contained in the FC subgroup of G and $G = < T, g >$.

(iv) $G = T \rtimes X$ where T is finite Abelian and X is torsion free Abelian.

Proof

(i) and (ii) follow immediately from Lemma 2.4 and Theorem 2 of [4].

Assume we are in case (iii) and $u \in \tilde{Z}(\mathcal{U})$. Since the conditons of Lemma 2.4 are satisfied, $u \in N_{\mathcal{U}}(G)$. But now Theorem 1.4 of [4] says that $u = g^i u_0$ where $u_0 \in \mathbb{Z}T$. Because $u_0 g u_0^{-1} \in G$ and $u_0 g u_0^{-1} - g \in [\mathbb{Z}G, \mathbb{Z}G], u_0 g u_0^{-1}$ and g must be conjugate in G and hence in T. We have $u_0 g u_0^{-1} = t_0^{-1} g t_0$ for some $t_0 \in T$. Hence $t_0 u_0 \in Z(\mathcal{U})$ and we have our result.

Finally assume we are in case (iv) and $u \in \tilde{Z}(\mathcal{U})$. Again Lemma 2.4 says $u \in N_{\mathcal{U}}(G)$. Theorem 1.4 in [4] tells us that $u = g u_0$ where $u_0 \in \mathbb{Z}T$ and $g \in G$. Let $H = C_G(T)$ and $K = H \cap X$. Observe that K is a central subgroup of G and that $\overline{G} = G/K \cong T \rtimes (X/K)$ is a finite metabelian group. Since $u_0 \in C_G(T)$, we can conclude from Corollary 2.6 of [6] that conjugation in $\overline{G}$ by $\overline{u_0}$ is an inner automorphism. It follows that $\overline{u_0 g_0^{-1}}$ is central in $\mathbb{Z}\overline{G}$ for some $g_0 \in G$.

We claim that $u_0 g_0^{-1}$ is a central unit in $\mathbb{Z}G$. Let $h \in G$. Since $u_0 g_0^{-1} \in N_{\mathcal{U}}(G)$, we have seen that $[u_0 g_0^{-1}, h] \in K$. However it is obvious that $[u_0 g_0^{-1}, h] \in G'$. Since $G' \subseteq T$ and $T \cap K = 1$, our result follows.

□

Note that while cases (iii) and (iv) are examples of the family of groups discussed in Theorem 2.3, cases (i) and (ii) give additional information about the weaker result $\tilde{Z}(\mathcal{U}) \subseteq G \cdot C_{\mathcal{U}}(T)$ as well. We conjecture that $\tilde{Z}(\mathcal{U}) \subseteq G \cdot C_{\mathcal{U}}(T)$ holds whenever G is an FC group.

The next lemma is needed to prove Theorem 2.7.

Lemma 2.6

(a) Assume $< t > \triangleleft G$ for any torsion element t of G. Then if u is a hypercentral unit in $\mathbb{Z}G, utu^{-1} = t$ or t^{-1} for any torsion element t of G. Moreover, if $utu^{-1} = t^{-1}$ then the order of t is a power of 2.

(b) Assume $< t > \lhd G$ for any torsion element t of G. Assume as well that the torsion elements of G form an Abelian subgroup T of G. Then if u is a hypercentral unit in $\mathbb{Z}G$, either $utu^{-1} = t$ for all $t \in T$ or $utu^{-1} = t^{-1}$ for all $t \in T$. In the latter case, T is a 2-group.

Proof.

(a) Note that the given condition implies that the torsion elements of G form a subgroup T of G.

Say $u \in Z_n(\mathcal{U})$ and $t \in T$. By Lemma 2.2 we know that $utu^{-1} \in T$. Since $\overline{utu^{-1}} = \overline{1}$ in $\mathbb{Z}(G/ < t >)$, we conclude further that $utu^{-1} = t^i$ for some integer $i, 1 \leq i \leq o(t) - 1$. For convenience we set $l = \phi(o(t))$, where ϕ denotes the Euler phi function, in the rest of this proof.

Say $t^i \neq t, t^{-1}$. In that case

$$b = (1 + t + \cdots + t^{i-1})^l + \frac{1 - i^l}{o(t)}\hat{t}$$

is a nontrivial Bass cyclic unit ([9], p.34) in $\mathbb{Z} < t > \subseteq \mathbb{Z}G$, and b is of infinite order.

Note $b^u = (1 + t^i + \cdots + t^{i(i-1)})^l + \frac{1 - i^l}{o(t)}\hat{t}$. In general,

$$b^{u^r} = (1 + t^{i^r} + \cdots + t^{i^r(i-1)})^l + \frac{1 - i^l}{o(t)}\hat{t}$$

for any r.

It follows that $bb^u \cdots b^{u^{l-1}}$ is equal to $(1 + t + \cdots + t^{i-1})^l (1 + t^i + \cdots + t^{i(i-1)})^l \cdots (1 + t^{i^{l-1}} + \cdots + t^{i^{l-1}(i-1)})^l + m\hat{t}$ for some integer m, and this product is equal to $(1 + t + t^2 + \cdots + t^{i^l - 1})^l + m\hat{t} = 1 + m_1\hat{t}$ for some integer m_1 (since $i^l \equiv 1 (\text{mod } o(t))$). Since b has augmentation 1, we conclude that $m_1 = 0$ and so

$$bb^u \cdots b^{u^{l-1}} = 1.$$

Now $u \in Z_n(\mathcal{U})$, so $b^{u^r} \equiv b (\text{mod } Z_{n-1}(\mathcal{U}))$ for all r, and we conclude that $b^l \in Z_{n-1}(\mathcal{U})$.

Next observe that

$$1 = (bb^u \cdots b^{u^{l-1}})^l = b^l (b^l)^u \cdots (b^l)^{u^{l-1}}$$

since all b^{u^r} are in $\mathbb{Z} < t >$ and thus they commute with one another. Since $b^l \in Z_{n-1}(\mathcal{U})$, we conclude as above that $b^{l^2} \in Z_{n-2}(\mathcal{U})$. Continuing we eventually get that $b^s = 1$ for some $s \geq 1$, contradicting the fact that b is of infinite order.

So $t^i = t$ or t^{-1} as desired.

For the second part, observe that if $u \in Z_n(\mathcal{U})$ and $utu^{-1} = t^{-1}$ then $t^2 \in Z_{n-1}(\mathcal{U})$. Since $ut^2u^{-1} = t^{-2}, t^4 \in Z_{n-2}(\mathcal{U})$. This process continues and gives the result.

(b) From (a) we know that if u is a hypercentral unit and $t \in T$ then $utu^{-1} = t$ or t^{-1}. Say we have $ut_1u^{-1} = t_1$ and $ut_2u^{-1} = t_2^{-1}$ for some $t_1, t_2 \in T$ where $t_1 \neq t_1^{-1}$ and $t_2 \neq t_2^{-1}$. We know $ut_1t_2u^{-1} = t_1t_2^{-1}$. But we also must have $ut_1t_2u^{-1} = t_1t_2$ or $(t_1t_2)^{-1}$, and $t_1t_2^{-1} = t_1t_2$ and $t_1t_2^{-1} = t_1^{-1}t_2^{-1}$ both lead to a contradiction. Now the second part follows immediately from (a).

□

As mentioned in the introduction, when G is torsion a complete description of $Z_n(\mathcal{U}) \cap G$ has been obtained for all n (of course $Z_n(\mathcal{U}) = Z_2(\mathcal{U}) = \tilde{Z}(\mathcal{U})$ for all $n \geq 2$ in this case). In general, if we are fortunate enough to have $\mathcal{U}(\mathbb{Z}G) = G \cdot Z(\mathcal{U})$ (as is the case when $\mathcal{U}$ is nilpotent, see Theorem 6.3.23 in [8]) then it is easy to see that $Z_n(\mathcal{U}) = Z_n(G) \cdot Z(\mathcal{U})$ for all $n > 1$. We also have

Theorem 2.7

Assume that the torsion elements of G form an Abelian subgroup T. In addition, assume every finite subgroup of G is normal in G and $G =< T, g >$.

(i) If $gtg^{-1} = t^{\pm 1}$ for all $t \in T$, then $Z_n(\mathcal{U}) = Z_n(G) \cdot Z(\mathcal{U})$ for all $n \geq 1$.

(ii) In all other cases, $Z_n(\mathcal{U}) = (T \cap Z_n(G)) \cdot Z(\mathcal{U})$ for all $n \geq 1$.

Proof

Note that the given conditions imply that $\mathcal{U}(\mathbb{Z}G) = G \cdot \mathcal{U}(\mathbb{Z}T)$ ([8], Chapter VI).

Case (i) follows from above because under our assumption any unit u in $\mathbb{Z}T$ can be written as $\alpha_1 t$ where α_1 is central ([9], p.10), and thus $\mathcal{U}(\mathbb{Z}G) = G \cdot Z(\mathcal{U})$.

Now assume we are in case (ii). We proceed by induction on n. If $n = 1$ we're done, so assume the result holds for $n = k, k \geq 1$.

If $t \in T \cap Z_{k+1}(G)$, then $tgt^{-1} = gt^j$ (since $< t > \triangleleft G$) where $t^j \in T \cap Z_k(G)$. The induction hypothesis says that $t^j \in Z_k(\mathcal{U})$ and, using $\mathcal{U}(\mathbb{Z}G) = G \cdot \mathcal{U}(\mathbb{Z}T)$, we conclude that $t \in Z_{k+1}(\mathcal{U})$, giving containment in one direction.

So now consider the other containment. Proposition 2.5 tells us that $Z_{k+1}(\mathcal{U}) \subseteq G \cdot Z(\mathcal{U})$, and the proof will be completed if we show that the first term in the product on the right can be chosen in $T \cap Z_{k+1}(\mathcal{U})$, since it is easy to see that $G \cap Z_n(\mathcal{U}) \subseteq Z_n(G)$ for all $n \geq 1$. To this end, assume a typical group element $t_1 g^k, t_1 \in T$, belongs to $G \cap Z_{k+1}(\mathcal{U})$. It follows from Lemma 2.6(b) that $(t_1 g^k)^{-1} t (t_1 g^k) = g^{-k} t g^k$ equals t or t^{-1} for all $t \in T$. If $g^{-k} t g^k = t$ were true for all t then g^k would be central and we would be done. So we can assume that $t_1 g^k \in G \cap Z_{k+1}(\mathcal{U})$ and $g^{-k} t g^k = t^{-1}$ for all $t \in T$ (and also T is a 2-group).

Since we are not in case (i), there must exist $t_0 \in T$ such that $g t_0 g^{-1} = t_0^l$ where l is odd, $1 < l < o(t_0) - 1$. It follows that $o(t_0) = 2^m$ for some $m \geq 3$.

Also $g t_0 g^{-1} = t_0^l$ and $g^k t_0 g^{-k} = t_0^{-1}$ imply that k must be even (the order of g as an automorphism of $< t_0 >$ must divide $\phi(2^m) = 2^{m-1}$ and be ≥ 4, while $2k$ is divisible by this order). Now $g^k t_0 g^{-k} = t_0^{l^k}$, so 2^m divides $1 + l^k$. But $l^k \equiv 1 (\mathrm{mod}\ 4)$, contradicting the fact that $o(t_0)$ is divisible by 4. We conclude that this situation cannot occur, and the proof is complete.

□

3 Second Centre

All results so far have concerned $Z_n(\mathcal{U})$ where n is any natural number. As mentioned earlier, when G is torsion this is equivalent to proving results about $Z_2(\mathcal{U})$, but this is not true in general. Some of the observations made earlier can be sharpened in the particular case $n = 2$. Our main result is Theorem 3.5 which characterizes T whenever $Z_2(\mathcal{U})$ is not contained in $C_{\mathcal{U}}(T)$.

The first result demonstrates a sharpening of Lemma 2.6.

Lemma 3.1

(a) If $t \in G$ is of finite order and $u \in Z_2(\mathcal{U})$, then $utu^{-1} = t$ or t^{-1}. Moreover, if $utu^{-1} = t^{-1}$ then the order of t divides 4.

(b) Assume the torsion elements of G form an Abelian subgroup T of G. Then if $u \in Z_2(\mathcal{U})$, either $utu^{-1} = t$ for all $t \in G$ or $utu^{-1} = t^{-1}$ for all $t \in T$. In the latter case, the exponent of T divides 4.

Proof.

(a) Let u, t be as stated, and assume $t^n = 1$. Note that $utu^{-1} = tz$ where $z \in Z(\mathcal{U})$, and $t^n = 1$ implies $z^n = 1$ also, so $z \in T$ and u^n commutes with t.

Consider the unipotent unit $v = 1 + (1 - t)u\hat{t}$. We know that u^n commutes with v. On the other hand $uv = vuc$ for some $c \in Z(\mathcal{U})$, so $c^n = 1$. It follows that u commutes with v^n. But $v^n = 1 + n(1 - t)u\hat{t}$, and we conclude that u commutes with v.

So $u(1 - t)u\hat{t} = (1 - t)u\hat{t}u$, or $(1 - t)u\hat{t}u^{-1} = (1 - u^{-1}tu)\hat{t}$.

Earlier in the proof we observed that utu^{-1} and $u^{-1}tu$ are both in T. It follows that $utu^{-1} \in < t >$.

Now exactly the same argument as in the proof of Lemma 2.6(a) can be used. If $utu^{-1} = t^i \neq t, t^{-1}$, we can construct a nontrivial Bass cyclic unit

$$b = (1 + t \cdots + t^{i-1})^{\phi(o(t))} + \frac{1 - i^{\phi(o(t))}}{o(t)}\hat{t}.$$

We have $bb^u \cdots b^{u^{\phi(o(t))-1}} = 1$, and this leads to the contradiction that b is of finite order.

So $utu^{-1} = t$ or t^{-1} as desired. The last statement follows from $utu^{-1} = t(t^{-2})$, which forces t^2 to be central.

(b) This follows in exactly the same way as in the proof of Lemma 2.6(b).

□

We require the following observation concerning unipotent units.

Lemma 3.2

Assume $t \in G$ is of finite order and $u \in Z_2(\mathcal{U})$. Then for any $\alpha \in \mathbb{Z}G, u$ commutes with the unipotent unit $v = 1 + (1-t)\alpha\hat{t}$.

Proof.

We know $uv = vuc$ for some $c \in Z(\mathcal{U})$. So $uv^2 = v^2uc^2$.

But $v^2 = 1 + 2(1-t)\alpha\hat{t} = 2v - 1$. Substituting we get $u(2v-1)u^{-1} = (2v-1)c^2$, or $2vc - 1 = 2vc^2 - c^2$. This means that

$$2(1-v)(c^2-c) = 2(c^2-c) + (1-c^2) = (c-1)^2.$$

Note that $(1-v)^2 = 0$, so we get $(c-1)^4 = 0$. Since $\mathbb{Z}G$ contains no nonzero central nilpotent elements, this gives $c = 1$ and we're done.

□

The following technical lemma will be needed in what follows.

Lemma 3.3

Assume that the set of torsion elements of G is a subgroup T of G and $< t_1 > \ntriangleleft T$ for some $t_1 \in T$. Then $[t,u] \in < t_1 >$ for all $u \in Z_2(\mathcal{U})$ and all $t \in T$. In addition, either $[t,u] = 1$ or $[t,u]$ is the unique (central) element of order 2 in $< t_1 >$ and, in the latter case, $t^2 = [t,u]$.

Proof.

Choose $u \in Z_2(\mathcal{U})$ and $t \in T$.

First assume that $tt_1t^{-1} \notin < t_1 >$. In that case, $v = 1 + (1-t_1)t^{-1}\hat{t_1}$ is a nontrivial bicyclic unit and Lemma 3.2 tells us that $uvu^{-1} = v$. So $1 + (1 - ut_1u^{-1})(ut^{-1}u^{-1})\hat{t_1} = 1 + (1-t_1)t^{-1}\hat{t_1}$. Again using Lemma 3.1, we conclude that either $[t,u] = 1$ or $utu^{-1} = t^{-1}$ and, in the latter case, $t = t^{-1}t_1^i$ for some i and $[t,u] = t^2 = t_1^i$ is central of order 2.

Next assume that $tt_1t^{-1} \in< t_1 >$. We know there exists $t_2 \in T$ such that $t_2t_1t_2^{-1} \notin< t_1 >$ and the above paragraph tells us that $[t_2, u] \in< t_1 >$. Similarly $(t_2t)t_1(t_2t)^{-1} \notin< t_1 >$ implies $[t_2t, u] \in< t_1 >$. Since $[t_2, u]$ and $[t_2t, u]$ are both central, $[t, u] \in< t_1 >$ also and the other results follow easily.

□

One of the questions completely settled for torsion groups in [7] is that of determining when $Z_2(\mathcal{U}) \neq Z(\mathcal{U})$. Information on this for more general groups can be deduced from the earlier results as well as the following example, in which a family of groups with $Z_2(\mathcal{U}) \neq Z(\mathcal{U})$ are constructed.

Example 3.4

Let A be an Abelian group containing an element h of order 4. Let $G =< A, x|xax^{-1} = a^{-1}$ for all $a \in A, x^2 = h^2 >$, i.e. G is like a Q^*-group but may contain elements of infinite order. Then the argument outlined in the first paragraph of the proof of Theorem 2 in [7] shows that $h \in Z_2(\mathcal{U})$ but $h \notin Z(\mathcal{U})$.

□

We will close with our main result, which deals with the weaker question of when $Z_2(\mathcal{U}) \nsubseteq C_{\mathcal{U}}(T)$.

Theorem 3.5

Assume that the set of torsion elements of G is a subgroup T of G. If $Z_2(\mathcal{U}) \nsubseteq C_{\mathcal{U}}(T)$, then T is either an Abelian group of exponent 4 or a Q^*-group.

Proof.

If T is Abelian, we know from Lemma 3.1(b) that it must be of exponent 4.

Assume T is Dedekind group, i.e. $T \cong K_8 \times E_2 \times E_2^1$ where E_2 is an elementary Abelian 2-group and E_2^1 is a group of odd order. Let $u \in Z_2(\mathcal{U})\backslash C_{\mathcal{U}}(T)$. Then Lemma 3.1(a) says $utu^{-1} = t$ for all $t \in E_2 \times E_2^1$, so there exists $x \in K_8$ of order 4 such that $uxu^{-1} = x^3$. Say it is possible to choose $y \neq 1$ in E_2^1 of prime order p. Since xy is of order $4p, uxyu^{-1} = xy$. But $uxyu^{-1} = x^3y$. This contradiction tells us that $T \cong K_8 \times E_2$, a Hamiltonian 2-group, and this is a Q^*-group.

So assume now that there exists $t_1 \in T$ with $< t_1 > \ntriangleleft T$. Again let $u \in Z_2(\mathcal{U}) \setminus C_{\mathcal{U}}(T)$. Let $H = \{t \in T | [t, u] = 1\}$. We know $H \neq T$ and, using Proposition 4.1(a), it is easy to see that $H \triangleleft T$. Moreover, if $x, y \in T \setminus H$ then $[x, u] = [y, u]$ is central of order 2 by Lemma 3.3, so $[xy, u] = 1$ and $xy \in H$. Thus $|T/H| = 2$.

If $a \in T \setminus H$ and $x \in H$, then $a^2 = (ax)^2$ by Lemma 3.3 so $axa^{-1} = x^{-1}$. If in addition $y \in H$ then $aya^{-1} = y^{-1}$ and $axya^{-1} = (xy)^{-1} = y^{-1}x^{-1}$. So $x^{-1}y^{-1} = y^{-1}x^{-1}$ and H is Abelian.

Finally, observing that T/H^2 is Abelian we conclude that if $[t, u] \neq 1$ then $[t, u] \in H^2$. This completes the proof that T is a Q^*-group.

□

It is easy to see that both possibilities can occur. If G is a Q^*-group, then $Z_2(\mathcal{U}) \neq Z(\mathcal{U})$ ([7] or Example 3.4) and also $G = T$, so $Z_2(\mathcal{U}) \nsubseteq C_{\mathcal{U}}(T)$. Also, if $G =< t, g | t^4 = 1, gtg^{-1} = t^3 >$ then Theorem 2.7 says that $t \in Z_2(\mathcal{U})$.

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