THE NORMALIZER CONJECTURE IN THE ALTERNATIVE CASE

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ABSTRACT. Let L be a torsion loop for which the integral loop ring $\mathsf{Z}L$ is an alternative, but not associative, ring. Let $\mathcal{N}_{\mathcal{U}}$ denote the normalizer of L in the unit loop $\mathcal{U}(\mathsf{Z}L)$. We show that $\mathcal{N}_{\mathcal{U}}(L) = \mathcal{Z}(\mathcal{U})L$, $\mathcal{Z}(\mathcal{U})$ the centre of $\mathcal{U}(\mathsf{Z}L)$, and use this fact to show that $\mathcal{U}(\mathsf{Z}L)$ has central height 1, unless L is a hamiltonian 2-loop.

1. INTRODUCTION

A Moufang loop is an *RA loop* if the loop ring RL is alternative, but not associative, for any commutative associative ring R with unity [2, Corollary IV.1.2]. That there are such loops was first noticed in 1983 [3]. By now, the theory of RA loops is well developed and described in the monograph [2], which is the primary reference for this paper.

If L is a loop, the integral loop ring of L is denoted $\mathsf{Z}L$. If $\mathsf{Z}L$ is an alternative ring, the invertible elements or *units* in $\mathsf{Z}L$ form a Moufang loop which we denote $\mathcal{U}(\mathsf{Z}L)$. This loop contains L as a subloop, but rarely as a normal subloop. Indeed, the first author and Polcino Milies have shown recently that an RA loop L is normal in $\mathcal{U}(\mathsf{Z}L)$ if and only if $\mathcal{U}(\mathsf{Z}L) = \pm L$ [4]. In this paper, we turn to a related question and ask what is the largest subloop of $\mathcal{U}(\mathsf{Z}L)$ in which L is normal; that is, what is the *normalizer* of L in $\mathcal{U}(\mathsf{Z}L)$?

Denote this normalizer $\mathcal{N}_{\mathcal{U}}(L)$ and let $\mathcal{Z}(\mathcal{U})$ denote the centre of $\mathcal{U}(\mathsf{Z}L)$. Clearly L is normal in $\mathcal{Z}(\mathcal{U})L$, so $\mathcal{Z}(\mathcal{U})L \subseteq \mathcal{N}_{\mathcal{U}}(L)$. The *normalizer conjecture* says that there is equality here, that is, $\mathcal{N}_{\mathcal{U}}(L) = \mathcal{Z}(\mathcal{U})L$. In this paper, the conjecture is established in the case that L is a torsion RA loop.

While the normalizer conjecture in the associative case is false in general [5], it is true for various classes of groups and a property of interest to people who study group rings. It was proven first for finite nilpotent groups by D. B. Coleman [1] and later, for finite groups which have a normal Sylow 2-subgroup by S. Jackowski and Z. Marciniak [6]. Recently, the second author, together with M. M. Parmenter and S. K. Sehgal, obtained the result for finite groups whose nonnormal subgroups have nontrivial intersection [7].

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Section I.1 and Chapter II of [2] provide more than enough background in alternative rings and loops for the purposes of this paper. Here, we present briefly some ideas of particular importance. An alternative ring is a ring which satisfies both identities

$$x(xy) = x^2y, \quad (yx)x = yx^2.$$

A Moufang loop is a loop which satisfies any of the three equivalent identities

$$x(y \cdot xz) = (xy \cdot x)z, \quad x(y \cdot zy) = (xy \cdot z)y, \quad (xy)(zx) = (x \cdot yz)x.$$

The centre of an alternative ring or a Moufang loop is the set of elements which commute with all other elements and associate with all other pairs of elements. An alternative ring or an Moufang loop is *diassociative* in the sense that the subring (or subloop) generated by any two elements is associative. Consequently, nonassociative products involving just two elements and perhaps additional elements from the centre do not require the insertion of parentheses to indicate order of multiplication.

Let L be a Moufang loop. For $x \in L$, the right and left translation maps $R(x), L(x) \colon L \to L$ are defined by

$$aR(x) = ax, \quad aL(x) = xa$$

for $a \in L$. For $x, y \in L$, define *inner maps* $T(x)$, $R(x, y)$ and $L(x, y)$ by

$$\begin{aligned} T(x) &= R(x)L(x)^{-1} \\ R(x,y) &= R(x)R(y)R(xy)^{-1} \\ L(x,y) &= L(x)L(y)L(yx)^{-1}. \end{aligned}$$

Each of these maps is a semiautomorphism θ, that is, $(xyx)\theta = x\theta y\theta x\theta$ for all $x, y \in L$ [2, Theorem II.3.3], and it follows that

$$(x^n)\theta = (x\theta)^n \tag{1.1}$$

for any $x \in L$ and any integer n. A subloop N of L is *normal* if $N\theta \subseteq N$ for any inner map θ. In a Moufang loop, $L(x, y) = R(x^{-1}, y^{-1})$ [2, Theorem II.3.3], so one only has to check closure under maps of the form $T(x)$ and $R(x, y)$ to verify normality.

2. The Normalizer Conjecture

Let $\alpha = \sum_{\ell \in L} \alpha_\ell \ell$ be an element of a loop ring RL. The *support* of α is the set $\operatorname{supp}(\alpha)$ of loop elements which actually appear in this representation of α:

$$\operatorname{supp}(\alpha) = \{\ell \mid \alpha_\ell \neq 0\}.$$

By definition of *loop ring*, all elements of RL have finite support. The *augmentation* of α is the element $\epsilon(\alpha) = \sum \alpha_\ell \in R$. If $\ell \in L$ has finite order n, we shall use the notation $\widehat{\ell}$ for the element $1 + \ell + \ell^2 + \cdots + \ell^{n-1}$ of RL:

$$\widehat{\ell} = 1 + \ell + \ell^2 + \cdots + \ell^{n-1}.$$

Lemma 2.1. *Let RL be an alternative loop ring, let $\alpha \in RL$ and suppose $\ell\alpha = \alpha$ for some $\ell \in L$. Then ℓ has finite order n and there exists $\beta \in RL$ such that $\alpha = \widehat{\ell}\beta$.*

Proof. If $g \in \operatorname{supp}(\alpha)$, then $\ell^i g \in \operatorname{supp}(\alpha)$ for all i. Since $\operatorname{supp}(\alpha)$ is finite, ℓ has finite order, say n, and $\alpha = \widehat{\ell}\beta_1 + \beta_2$ for $\beta_1, \beta_2 \in RL$. Since $\ell\widehat{\ell} = \widehat{\ell}$, so also $\ell\beta_2 = \beta_2$, so the result follows by induction on $|\operatorname{supp}(\alpha)|$. □

Theorem 2.2. *Let L be a torsion RA loop, let $\mathcal{Z}(\mathcal{U})$ denote the centre of the unit loop $\mathcal{U}(\mathsf{Z}L)$ and let $\mathcal{N}_{\mathcal{U}}(L)$ denote the normalizer of L in $\mathcal{U}(\mathsf{Z}L)$. Then $\mathcal{N}_{\mathcal{U}}(L) = \mathcal{Z}(\mathcal{U})L$.*

Proof. Let A denote the centre of L. It is known that L/A is a finite elementary abelian 2-group [2, Corollary IV.2.3]. Thus we may write $L = \cup_{x_i \in \mathcal{T}} Ax_i$ as the disjoint union of cosets of A and, without loss of generality, assume that $x_1 = 1$. Let $u \in \mathcal{N}_{\mathcal{U}}(L)$. Then we can write

$$u = \sum_{x_i \in \mathcal{T}} \alpha_i x_i \tag{2.1}$$

with $\alpha_i \in \mathsf{Z}A$. Letting $\alpha \mapsto \overline{\alpha}$ denote the extension to $\mathsf{Z}L$ of the natural homomorphism $L \to L/A$, we have $\overline{u} = \sum \overline{\alpha}_i \overline{x}_i = \sum \epsilon(\alpha_i)\overline{x}_i \in \mathsf{Z}[L/A]$. By a theorem of G. Higman, the abelian group ring $\mathsf{Z}[L/A]$ has only trivial units [2, Theorem VIII.3.1], so $\epsilon(\alpha_i) = \pm 1$ for precisely one i, while $\epsilon(\alpha_j) = 0$ for $j \neq i$. Replacing u by $\pm x_i^{-1}u$, it suffices to prove that a unit of the form (2.1) with $\epsilon(\alpha_1) = 1$ and $\epsilon(\alpha_i) = 0$ for $i \neq 1$ is central in $\mathcal{U}(\mathsf{Z}L)$.

Let L' denote the commutator/associator subloop of L. It is known that $L' = \{1, s\}$ for some central element s necessarily of order 2 [2, Theorem IV.1.8]. For any $\ell \in L$, the commutator $(u, \ell) = u^{-1}\ell^{-1}u\ell = \ell^{-1}T(u) \cdot \ell \in L$ and the image of this element in $\mathsf{Z}[L/L']$ is the image of 1. It follows that $(u, \ell) \in L' = \{1, s\}$. We claim that $(u, \ell) = 1$ for any ℓ. If this is not the case, then there exists $\ell \in L$ such that $u\ell = s\ell u$. Hence,

$$\sum_{x_i \in \mathcal{T}} \alpha_i x_i \ell = s \sum_{x_i \in \mathcal{T}} \alpha_i \ell x_i = \sum_{x_i \in \mathcal{T}} s\alpha_i(\ell, x_i)x_i\ell.$$

Since the commutators (ℓ, x_i) are central, we obtain

$$\sum_{x_i \in \mathcal{T}} \alpha_i x_i = \sum_{x_i \in \mathcal{T}} s\alpha_i(\ell, x_i)x_i.$$

Comparing coefficients of $x_1 = 1$ on both sides of this equation gives $\alpha_1 = s\alpha_1$ and hence, by Lemma 2.1, that $\alpha_1 = \widehat{s}\beta$ for some $\beta \in \mathsf{Z}L$ with $\beta \neq 0$ because $\epsilon(\alpha_1) = 1$. Taking augmentations gives the contradiction $1 = 2\epsilon(\beta)$.

It follows that u commutes with every element of L. In the loop ring of an RA loop, this implies that u also associates with every pair of elements of L and hence is central [2, Corollary III.4.2]. □

If L is a hamiltonian 2-loop, the unit loop of $\mathsf{Z}L$ is just $\pm L$ [2, Theorem VIII.3.1] and of course L is normal in $\mathcal{U}(\mathsf{Z}L)$. Conversely, suppose that the torsion RA loop L is normal in $\mathcal{U}(\mathsf{Z}L)$. Then $\mathcal{U}(\mathsf{Z}L) = \mathcal{N}_{\mathcal{U}}(L) = \mathcal{Z}(\mathcal{U})L$. It follows easily that the commutator of two units and the associator of three units are a commutator and an associator, respectively, in L. Thus the commutator/associator subloop $[\mathcal{U}(\mathsf{Z}L)]'$

is the group L', which has order 2. This forces L to be a hamiltonian 2-loop [2, Corollary XII.2.14] and strengthens a result known previously in the case of finite loops [4, Theorem 3.1].

Corollary 2.3. *A torsion RA loop L is normal in the unit loop of $\mathsf{Z}L$ if and only if L is a hamiltonian 2-loop.*

3. Central Height

Just as in group theory, a Moufang loop L has an *upper central series*

$$\{1\} = \mathcal{Z}_0 \subseteq \mathcal{Z}_1 \subseteq \mathcal{Z}_2 \subseteq \cdots$$

with $\mathcal{Z}_{i+1}/\mathcal{Z}_i = \mathcal{Z}(L/\mathcal{Z}_i)$. If this series terminates at L in a finite number of steps, then L is called *nilpotent*. If L is an RA loop, it is known that $\mathcal{U}(\mathsf{Z}L)$ is nilpotent if and only if L is a hamiltonian 2-loop which is not associative [2, Corollary XII.2.14]. In this section, we show just how far from nilpotency is $\mathcal{U}(\mathsf{Z}L)$ in the case that L is not a hamiltonian 2-loop. Our main result (see Theorem 3.3) is that if L is an RA loop which is not a hamiltonian 2-loop, then $\mathcal{U}(\mathsf{Z}L)$ has *central height* 1, that is, $\mathcal{Z}_2(\mathcal{U}(\mathsf{Z}L)) = \mathcal{Z}_1(\mathcal{U}(\mathsf{Z}L)) = \mathcal{Z}(\mathcal{U}(\mathsf{Z}L))$.

Lemma 3.1. *Let L be a torsion RA loop. If L contains a noncentral element ℓ_0 such that $\langle \ell_0 \rangle$ is normal in $\mathcal{U}(\mathsf{Z}L)$, then L is a hamiltonian 2-loop.*

Proof. Choose $x \in L$ such that $x\ell_0 = s\ell_0 x$. Then $s\ell_0 = x\ell_0 x^{-1} \in \langle \ell_0 \rangle$ implies $s \in \langle \ell_0 \rangle$. Let u be any unit of $\mathsf{Z}L$. The commutator $(u, \ell_0) = (u^{-1}\ell_0^{-1}u)\ell_0$ is in L and its image in the abelian group ring $\mathsf{Z}[L/L']$ is the image of 1, so $(u, \ell_0) \in L'$. Thus

$$(u, \ell_0) \in L' = \{1, s\} \in \langle \ell_0 \rangle \tag{3.1}$$

and so

$$uT(\ell_0) = \ell_0^{-1} u \ell_0 = u(u, \ell_0) \in L'u.$$

Since $T(\ell_0)$ is a semiautomorphism and $uT(\ell_0) = cu$ for some $c \in L'$, we have $u^2 T(\ell_0) = (uT(\ell_0))^2 = c^2u^2 = u^2$ by (1.1), so

$$(u^2, \ell_0) = 1. \tag{3.2}$$

Let $x, y \in L$ with x of finite order. The element $r = (1 - x)y\widehat{x}$ satisfies $r^2 = 0$, so $v = 1 + r$ is a unit (with inverse $1 - r$), a so-called *bicyclic unit*, (See [2, §VIII.2].) Now $v^2 = 1 + 2r$ and $v^2\ell_0 = \ell_0 v^2$ by (3.2), so $r\ell_0 = \ell_0 r$. Hence

$$(v, \ell_0) = 1. \tag{3.3}$$

In an RA loop, a subloop is normal if and only if it is central or it contains s [2, Corollary IV.1.11].

Suppose there exists $\ell \in L$ with $\langle \ell \rangle$ not normal. Suppose also that $\ell\ell_0 \neq \ell_0\ell$ (thus $\ell_0\ell = s\ell\ell_0$) and consider the bicyclic unit

$$v = 1 + (1 - \ell)\ell_0\widehat{\ell} = 1 + \ell_0(1 - s\ell)\widehat{\ell} = 1 + \ell_0(1 - s)\widehat{\ell}.$$

By (3.3), $\ell_0^2(1-s)\widehat{\ell} = \ell_0(1-s)\widehat{\ell}\ell_0$, that is,

$$\ell_0(1-s)\widehat{\ell} = (1-s)\widehat{\ell}\ell_0.$$

This gives $\ell_0\ell = \ell^i\ell_0$ for some i, so $s\ell\ell_0 = \ell^i\ell_0$ and $s = \ell^{i-1} \in \langle \ell \rangle$ contradicting the fact that $\langle \ell \rangle$ is not normal in L. All this proves that if $\ell \in L$ and $\langle \ell \rangle$ is not normal in L, then ℓ and ℓ_0 must commute.

An RA loop M has the so-called *LC property*: $g, h \in M$ commute if and only if $g \in \mathcal{Z}(M)$ or $h \in \mathcal{Z}(M)$ or $g = zh$ for $z \in \mathcal{Z}(M)$ [2, §IV.2]. Here then, we have $\ell = z\ell_0$ for some $z \in \mathcal{Z}(L)$.

Since ℓ_0 is not central, there exists $x \in L$ with $x\ell_0 \neq \ell_0 x$, and hence $x\ell_0 = s\ell_0 x$. Consider the bicyclic unit

$$v = 1 + (1-\ell)x\widehat{\ell} = 1 + x(1-s)\widehat{\ell}.$$

Using (3.3), we have $\ell_0 x(1-s)\widehat{\ell} = x(1-s)\widehat{\ell}\ell_0$, so $sx\ell_0(1-s)\widehat{\ell} = x(1-s)\widehat{\ell}\ell_0$. Thus $-\ell_0(1-s)\widehat{\ell} = (1-s)\widehat{\ell}\ell_0$ and

$$2(1-s)\widehat{\ell}\ell_0 = 0 \tag{3.4}$$

because ℓ_0 and ℓ, and hence ℓ_0 and $\widehat{\ell}$, commute. As shown previously, an equation like (3.4) gives $\ell^i = s\ell$ for some i, so $s \in \langle \ell \rangle$, a contradiction.

We have shown that $\langle \ell \rangle$ is normal in L for any $\ell \in L$. It follows that L is hamiltonian, hence the direct product $\mathcal{C} \times E \times A$ of the Cayley loop, $\mathcal{C}$, an elementary abelian 2-group, E, and an abelian group, A, all of whose elements have odd order [2, Theorem II.4.8]. It remains to show that A is trivial.

Write $\ell_0 = cea$ with $c \in \mathcal{C}$, $e \in E$ and $a \in A$. Since ℓ_0 is not central, neither is c, so c has order 4. Let $n = \mathrm{o}(a)$, the order of a. Then $\ell_0^n = c^n e$ is not central. For any inner map θ, we have $(\ell_0^n)\theta = (\ell_0\theta)^n$ by (1.1); thus $\langle \ell_0^n \rangle$ is normal in $\mathcal{U}(\mathsf{Z}L)$. Replacing ℓ_0 by ℓ_0^n, without loss of generality, we may assume henceforth that ℓ_0 is an element of order 4 in $\mathcal{C} \times E$.

Assume A is not trivial and let $a \in A$ have (odd) prime order p. Choose $c \in \mathcal{C}$ with $(\ell_0, c) = s$. Then $g = ac = ca$ has order $4p$. Also $s = \ell_0^2 = c^2 = g^{2p}$ and

$$\ell_0^{-1}g\ell_0 = \ell_0^{-1}ca\ell_0 = \ell_0^{-1}c\ell_0 a = c^3 a = c^{2p+1}a^{2p+1} = g^{2p+1}.$$

If $p > 3$, we form the *Bass cyclic unit*

$$u = (1+g+g^2)^{2(p-1)} + \frac{1-3^{2(p-1)}}{4p}\widehat{g} \tag{3.5}$$

(see [2, VIII.2.3]). By (3.1), we have either $\ell_0^{-1}u\ell_0 = u$ or $\ell_0^{-1}u\ell_0 = us = ug^{2p}$. In the first case,

$$(1+g^{2p+1}+g^2)^{2(p-1)} = (1+g+g^2)^{2(p-1)},$$

which is impossible because the coefficient of g^{4p-1} is zero on the right hand side but nonzero on the left. In the second case,

$$(1+g^{2p+1}+g^2)^{2(p-1)} = (1+g+g^2)^{2(p-1)}g^{2p},$$

which is impossible because the coefficient of g^{2p} is 1 on the right but greater than 1 on the left. Thus $p = 3$.

With $p = 3$, we have $\ell_0^{-1} g \ell_0 = g^7$ and $\mathrm{o}(g) = 12$. This time, we consider the *Hoechsmann unit*

$$u = (1 + g^5 + g^{10} + g^3 + g^8)^2 - 2\widehat{g}$$

(see [8, p. 34]). As before, either $\ell_0^{-1} u \ell_0 = u$ or $\ell_0^{-1} u \ell_0 = su = ug^6$. In the first case, we have

$$(1 + g^{11} + g^{10} + g^9 + g^8)^2 = (1 + g^5 + g^{10} + g^3 + g^8)^2,$$

which is impossible because the coefficient of g is zero on the left hand side but nonzero on the right. In the second case, we have

$$(1 + g^{11} + g^{10} + g^9 + g^8)^2 = (1 + g^5 + g^{10} + g^3 + g^8)^2 g^6,$$

which is impossible because the coefficient of g^6 is 1 on the right but 3 on the left. It follows that A is trivial and the proof is complete. $\square$

Lemma 3.2. *Let L be a torsion RA loop. Let $\ell \in L$ be noncentral and suppose $\langle \ell \rangle$ is not normal in $\mathcal{U}(\mathsf{Z}L)$. Then $\ell \notin \mathcal{Z}_2(\mathcal{U})$.*

Proof. Assume $\ell \in \mathcal{Z}_2(\mathcal{U})$. First of all, we claim that $s \notin \langle \ell \rangle$. Let $u \in \mathcal{U}(\mathsf{Z}L)$. Then $\ell^{-1} u^{-1} \ell u = (\ell, u) = z \in \mathcal{Z}(\mathcal{U})$, so $\ell T(u) = u^{-1} \ell u = z\ell$. Thus $u^{-1} \ell u$ commutes with ℓ and both these elements have finite order. It follows that z is a central torsion unit and therefore *trivial* in the sense that $z \in \pm L$ [2, Corollary VIII.1.7]. Since the augmentation of (ℓ, u) is 1, in fact, $z \in +L$. Thus $(u, \ell) \in L$ and, modulo L', this element is 1, so $z = (u, \ell) \in L'$ and $\ell T(u) \in L' \langle \ell \rangle$.

Let $u, v \in \mathcal{U}(\mathsf{Z}L)$. Then $(\ell, u, v) = z' \in \mathcal{Z}(\mathcal{U})$, so $\ell u \cdot v = z' \ell \cdot uv$ and $\ell R(u, v) = (\ell u \cdot v)(uv)^{-1} = z' \ell$. Thus $\ell R(u, v)$ and ℓ are commuting elements of finite order (of the same order, in fact, as shown by (1.1)), so z' is a central torsion unit of $\mathsf{Z}L$ and hence an element of L. As before, $z' \in L'$. It follows that $\ell \theta \in L' \langle \ell \rangle$ for every inner map θ. Since $\langle \ell \rangle$ is not normal in $\mathcal{U}(\mathsf{Z}L)$, $L' \not\subseteq \langle \ell \rangle$, so $s \notin \langle \ell \rangle$, as claimed.

Since ℓ is not central, there exists $g \in L$ with $(g, \ell) = s$. Consider the bicyclic unit

$$u = 1 + (1 - \ell) g \widehat{\ell}.$$

Since $(u, \ell) \in L'$, $(u^2, \ell) = 1$ as in the proof of Lemma 3.1. Furthermore, since $u = 1 + r$ with $r^2 = 0$, it follows similarly that $(u, \ell) = 1$ and then that $s = \ell^i$ for some i. This contradiction to the assumption $\ell \in \mathcal{Z}_2(\mathcal{U})$ completes the proof. $\square$

Theorem 3.3. *Let L be a torsion RA loop. If L is not a hamiltonian 2-loop, then $\mathcal{Z}_2(\mathcal{U}) = \mathcal{Z}_1(\mathcal{U})$ $(= \mathcal{Z}(\mathcal{U}))$. If L is a hamiltonian 2-loop, $\mathcal{Z}_2 = \mathcal{U}(\mathsf{Z}L) = \pm L$.*

Proof. The second statement is just the assertion that in the hamiltonian 2-loop case, $\mathcal{U}(\mathsf{Z}L)$ is nilpotent of class 2. This is known [2, Corollary XII.2.14].

The proof of the first statement derives from the observation that L is normal in $\mathcal{Z}_2(\mathcal{U})L$ since $\ell T(u)$ and $\ell R(u, v)$ are both in $L' \langle \ell \rangle$ for any $u, v \in \mathcal{Z}_2$, as in the proof of Lemma 3.2. By Theorem 2.2, $\mathcal{Z}_2(\mathcal{U}) \subseteq \mathcal{Z}(\mathcal{U})L$. Now assume there exists $u \in \mathcal{Z}_2(\mathcal{U}) \setminus \mathcal{Z}_1(\mathcal{U})$. Write $u = z\ell$ with $z \in \mathcal{Z}(\mathcal{U})$ and $\ell \in L$. Then $\ell \in L \cap \mathcal{Z}_2(\mathcal{U})$ is not central. By Lemma 3.2, $\langle \ell \rangle$ is normal in $\mathcal{U}(\mathsf{Z}L)$. By Lemma 3.1, L is a hamiltonian 2-loop, a contradiction. $\square$

References

[1] Donald B. Coleman, *On the modular group ring of a p-group*, Proc. Amer. Math. Soc. **15** (1964), 511–514.
[2] E. G. Goodaire, E. Jespers, and C. Polcino Milies, *Alternative loop rings*, North-Holland Math. Studies, vol. 184, Elsevier, Amsterdam, 1996.
[3] Edgar G. Goodaire, *Alternative loop rings*, Publ. Math. Debrecen **30** (1983), 31–38.
[4] Edgar G. Goodaire and César Polcino Milies, *A normal complement for an RA loop in its integral loop ring*, to appear, 1999+.
[5] M. Hertweck, *A solution of the isomorphism problem*, preprint, 1997.
[6] S. Jackowski and Z. Marciniak, *Group automorphisms inducing the identity map on cohomology*, J. Pure Appl. Algebra **44** (1987), 241–250.
[7] Yuanlin Li, M. M. Parmenter, and S. K. Sehgal, *On the normalizer property for integral group rings*, Comm. Algebra **27** (1999), no. 9, 4217–4223.
[8] S. K. Sehgal, *Units in integral group rings*, Longman Scientific & Technical Press, Harlow, 1993.

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