On the normalizer problem for G-adapted group rings of torsion groups

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In this note, we prove that if a torsion group G has an abelian subgroup B such that G/B is abelian and R is a G-adapted ring with the property that $R(G/B)$ has only trivial units then G has the normalizer property in RG.

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1 Introduction and Preliminary

Let G be a torsion group. A ring R is called a G-adapted ring if R is an integral domain of characteristic 0 in which no prime dividing the order of any element of G is invertible. Throughout this article the coefficient ring R in the group ring RG is always assumed to be a G-adapted ring. Let $\mathcal{U}(RG)$ be the group of units of the G-adapted group ring RG and $N_{\mathcal{U}}(G)$ be the normalizer of G in $\mathcal{U}(RG)$. Clearly, $N_{\mathcal{U}}(G)$ contains G and also contains $\mathcal{Z} = \mathcal{Z}(\mathcal{U}(RG))$, the subgroup of central units of $\mathcal{U}$. Question 43 in [15] (i.e., the normalizer problem) asks whether $N_{\mathcal{U}}(G) = G\mathcal{Z}$ when G is finite. If the above equality holds, then we say that G has the normalizer property in RG. It was believed that the equality holds for all finite groups until recently

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Hertweck found counterexamples to the problem. Using them and a smart generalization of a result of Mazur [13], Hertweck constructed a counterexample to the well-known isomorphism problem [2]. Since the normalizer problem is intimately connected to the isomorphism problem it is very much of interest to know which groups enjoy the normalizer property. In the past few years, a great amount of work on the problem has been done by several authors (see [3, 4, 6, 8, 9, 10, 11, 12, 14]). In this paper we prove that if a torsion metabelian group G has an abelian subgroup B such that G/B is abelian and $R(G/B)$ has only trivial units, then G has the normalizer property in RG. This extends a result of [9] (Proposition 2.20) on a finite group basis G in its integral group ring, to any torsion group basis G in its G-adapted ring.

Every unit $u \in N_{\mathcal{U}}(G)$ induces an automorphism φ of G such that $\varphi_u(g) = ugu^{-1}$ for all $g \in G$. We now consider the subgroup $Aut_{\mathcal{U}}(G)$ formed by all such automorphisms and it is not hard to see that the normalizer problem described in [15] is equivalent to Question 3.7 in Jackowski and Marciniak [5]:

" Is $Aut_{\mathcal{U}}(G) = Inn(G)$ for all finite groups G?"

It is convenient to use this equivalent form to discuss the normalizer problem here and our notation follows that in [15].

Next we introduce some terminology and preliminary results.

Definition 1.1. *Let G be a torsion group. A subgroup P is called a Sylow p-subgroup of G for a prime number p, if P is a maximal p-subgroup of G.*

It is not hard to see that there exists a maximal p-subgroup of G by Zorn's Lemma. We remark that the Sylow theorems for finite groups are no longer true in this context. For example, not all Sylow p-subgroups are conjugates of one another. We need the following two results. A proof for the first can be found in [7] (1.D.4 Lemma) and the second is a special case of Theorem 9 of [14].

Lemma 1.2. *If B is a normal subgroup of a locally finite group G such that the quotient group G/B is a countable p-group for some prime p then there is a p-subgroup P of G with $BP = G$.*

Lemma 1.3. *Let G be a torsion group and P be any p-subgroup of G. For any $u \in N_{\mathcal{U}}(G)$, define $\varphi \in Aut(G)$ such that $\varphi(g) = ugu^{-1}$ for every $g \in G$ as before. Then restricted to the subgroup P, the automorphism φ becomes inner. Moreover, we have $\varphi|_P = conj(x_0)|_P$ for some $x_0 \in supp(u) \subset G$. In particular, if G is a p-group, then $Aut_{\mathcal{U}}(G) = Inn(G)$, so the normalizer property holds for G.*

We include a proof for completeness, and we note that the technique used in the following proof will be required later in the proof of Lemma 2.2.

Proof. Let $u = \sum u(x)x \in N_{\mathcal{U}}(G)$, where $u(x) \in R$ and $x \in supp(u)$. Without loss of generality, we may assume that the augmentation of u is 1. For every group element $g \in G$, $\varphi(g) = ugu^{-1}$ is also a group element. Rewrite $u = \varphi(g)ug^{-1}$, and hence $\sum u(x)x = \sum u(x)\varphi(g)xg^{-1}(*)$. This shows that $\varphi(g)xg^{-1}$ is in the support of u for all $g \in G$. Define a left group action σ_g of G on $supp(u)$ as follows: $\sigma_g(x) = \varphi(g)xg^{-1}$. It follows from $(*)$ that $u(x)$ is a constant on each orbit of x. Restricting the action to P, we have that the p-subgroup P acts on $supp(u)$, and thus the length of every orbit must be a p-power. It follows that

$$1 = \epsilon(u) = \sum c_i p^{l_i},$$

where ϵ is the augmentation map, p^{l_i} is the length of the orbit of x_i and $u(x_i) = c_i$. Since p is not invertible in R, the above equality shows that $p^{l_j} = 1$ for some j; that is to say there is a fixed point of this action, say x_0. Therefore, we have $\varphi(g)x_0g^{-1} = \sigma_g(x_0) = x_0$ for all $g \in P$. Consequently, $\varphi(g) = x_0gx_0^{-1}$, and thus $\varphi|_P = conj(x_0)|_P$. We are done. □

2 The Main Result

In this section, we extend our earlier result on a finite group basis G (Proposition 2.20 in [9]) to any torsion group basis G. The main result is as follows:

Theorem 2.1. *Let G be a torsion metabelian group and let B be an abelian normal subgroup of G for which the quotient group $A = G/B$ is abelian. If RA has only trivial units then G has the normalizer property in RG.*

We prove Theorem 2.1 by means of the following two lemmas.

Lemma 2.2. *Let G be a torsion metabelian group and let B be an abelian normal subgroup of G for which the quotient group $A = G/B$ is abelian. Let A_p be the Sylow p-subgroup of A and G_p be the pre-image of A_p in G. If RA has only trivial units, then for every φ in $Aut_{\mathcal{U}}(G)$ the restriction of φ to G_p is inner.*

Proof. Let $u \in N_{\mathcal{U}}(G)$ and $\epsilon(u) = 1$. Define φ to be the automorphism induced by u as before. We will show that the restriction of φ to G_p is inner.

First we show that the restriction of φ to B is inner. Write $u = \sum_{i=0}^{n} \alpha_i a_i$ where the cosets $a_i B$ are pairwise distinct, $a_0 = 1$, and $\alpha_i \in RB$. Since RA has only trivial units, we have that in RA, $\bar{u} = \sum_{i=0}^{n} \epsilon(\alpha_i)\bar{a}_i$ is trivial. Thus only one $\epsilon(\alpha_l) = 1$ and the others are zero. That is $\epsilon(\alpha_j) = 0$ for all $j \neq l$. Multiplying by $a_l^{-1}b^{-1}$ for any b in the support of α_l if necessary, we may assume that $\epsilon(\alpha_0) = 1$, $\epsilon(\alpha_i) = 0$ for all $i \neq 0$ and $1 \in supp(u)$. Now we show that $\varphi(b) = b$ for all $b \in B$ (or $\varphi|_B = id|_B$). Since B is a normal subgroup and $u \in N_{\mathcal{U}}(G)$, we have that $[u, b] = ubu^{-1}b^{-1} \in G$. By going mod B, we obtain that $[\bar{u}, \bar{b}] = \bar{1}$, and therefore, $[u, b] \in B$. Thus there exists a $b_0 \in B$ depending on b such that $ub = b_0bu$. This implies that $\alpha_0 b = b_0 b \alpha_0$, and therefore, $\alpha_0(1 - b_0) = 0$. This simply says that the order of b_0 divides $\epsilon(\alpha_0) = 1$, forcing $b_0 = 1$. Hence $[u, b] = 1$, and thus $\varphi(b) = b$ for all $b \in B$. We remark that because $1 \in supp(u)$, the support of u is contained in the FC-centre of G by Corollary 1 in [14]. Since G is locally finite, it follows that the support of u is contained in some finite normal subgroup H of G. Therefore, the support of α_0 is contained in the finite normal subgroup $H_1 = B \bigcap H$.

Next let P be any p-subgroup of G. We now show that the restriction of φ to P is a conjugation by a group element b_1, and moreover, $b_1 \in supp(\alpha_0) \subset B$. The first result follows from Lemma 1.3. To show the second part, we use the same trick as that used in the proof of Lemma 1.3. This time we need only define a group action of P on the support of α_0, and then the fixed point $b_1 \in supp(\alpha_0)$ will do the job. We observe that for every $p \in P, \varphi(p)p^{-1} = upu^{-1}p^{-1} \in B$ because in $RA, \overline{\varphi(p)p^{-1}} = \bar{1}$. Since $up = \varphi(p)u$ and $u = \sum_{i=0}^{n} \alpha_i a_i$, we have $\sum_{i=0}^{n} \alpha_i a_i p = \sum_{i=0}^{n} \varphi(p)\alpha_i a_i$. This shows that $\alpha_0 p = \varphi(p)\alpha_0$ and thus $\alpha_0 = \varphi(p)\alpha_0 p^{-1}$. Write $\alpha_0 = \sum \alpha_0(b)b$, where $b \in supp(\alpha_0)$, $\alpha_0(b) \in R$. It follows that $\sum \alpha_0(b)b = \sum \alpha_0(b)\varphi(p)bp^{-1}$. Now we can define a group action of P on $supp(\alpha_0)$, sending every element b to $\varphi(p)bp^{-1}(\in supp(\alpha_0))$. As before, we can show that there is a fixed point $b_1 \in supp(\alpha_0)$. Therefore, $\varphi(p) = conj(b_1)(p)$ for all $p \in P$. Since $b_1 \in B$ and B is abelian, we conclude that $conj(b_1)|_B = id|_B = \varphi|_B$, and therefore, $\varphi|_{BP} = conj(b_1)|_{BP}$.

Finally we prove that the restriction of φ to G_p is a conjugation by a group element b_p in the support of α_0. For every finite subgroup F of G_p, we can find a finite subgroup A_F of A_p such that F is contained in the pre-image of A_F in G. By Lemma 1.2, there exists a p-subgroup P of G such that this pre-image is equal to BP. It follows that there is a b_F in the support of α_0 such that φ acts on F as conjugation by b_F. Since the support of α_0 is finite, it is not hard to see that one can find a b_p which works for all F. In fact if this is not true, then for each b in the support of α_0 there is such a F_b so that φ is not a conjugation by b on it. Therefore, for the finite subgroup generated by the groups F_b for all $b \in supp(\alpha_0)$, there would be no b such that φ is a conjugation by b on it. This leads to a contradiction. Note that G_p is the union of its finite subgroups, so the restriction of φ to G_p is a conjugation by b_p. Without loss of generality, for a fixed p we may assume that $\varphi|_{G_p} = id|_{G_p}$. We note that the support of α_0 is still contained in H_1, and what we just proved implies that the order of φ is finite, so we may assume that φ has a prime power order in the sequel. $\square$

Lemma 2.3. *Let G be a torsion metabelian group and let B be an abelian normal subgroup of G for which the quotient group $A = G/B$ is an abelian group of exponent 6. If RA has only trivial units then G has the normalizer property in RG.*

Proof. Let $\varphi = \varphi_u \in Aut_{\mathcal{U}}(G)$ where $u = \sum_{i=0}^{n} \alpha_i a_i$ as before. We now show that φ is inner. Let $A = A_2 \times A_3$, and let G_p be the pre-image of A_p in G for $p = 2$ or 3 as before. As we mentioned earlier, we may assume that the order of φ is a power of p. If $p \neq 3$, by Lemma 2.2 we may further assume that $\varphi|_{G_2} = id|_{G_2}$, $\varphi|_{G_3} = conj(b_3)|_{G_3}$ for some $b_3 \in supp(\alpha_0) \subset H_1$, where H_1 is the same subgroup as that used in the proof of Lemma 2.2. Now we define a map δ from A to B as follows: $\delta(a) = \varphi(g)g^{-1}$ for every $a \in A$, where g is any pre-image of a in G. It is routine to check δ is well defined, $\delta(a) \in B$, and $\delta(a)$ is a 1−cocycle. We claim that $\delta^k(a) = \varphi^k(g)g^{-1}$. For $k = 1$, this is clear. By induction, we assume that $\delta^{k-1}(a) = \varphi^{k-1}(g)g^{-1}$. Then

$$\delta^k(a) = \delta^{k-1}(a)\delta(a) = \varphi^{k-1}(g)g^{-1}\varphi(g)g^{-1} = \varphi(\varphi^{k-1}(g)g^{-1}g)g^{-1} = \varphi^k(g)g^{-1}.$$

Since $o(\varphi)$, the order of φ, is a power of p and $\delta^{o(\varphi)}(a) = \varphi^{o(\varphi)}(g)g^{-1} = 1$, we conclude that the order of $[\delta]$ is a power of p. On the other hand, we will show that the order of $[\delta]$ is a power of 3. Therefore, $[\delta]$ is trivial. We first note that the restriction of $[\delta]$ to A_2 is trivial. This is because for all $a_2 \in A_2, \delta(a_2) = \varphi(g_2)g_2^{-1} = 1$, where $g_2 \in G_2$ is any pre-image of a_2 in G, so $\varphi(g_2) = g_2$ by the above assumption. Since H_1 is a normal subgroup and H_1 is contained in the abelian subgroup B, we can define a group action of A_3 on H_1 by conjugation (i.e., for every $a_3 \in A_3$, $h_1^{a_3} = h_1^{g_3} \forall h_1 \in H_1$, where g_3 is any pre-image of a_3 in G). Because H_1 is finite, some subgroup C of A_3 of finite index k acts trivially on H_1. This means that the pre-image of C in G centralizes H_1. It follows that for every $a_3 \in C$, $\delta(a_3) = \varphi(g_3)g_3^{-1} = b_3g_3b_3^{-1}g_3^{-1} = 1$, where b_3 is in the support of α_0 ($\subset H_1$) as before. So the restriction of $[\delta]$ to C is also trivial. Now write $A_3 = C \times C'$, where the order of C', $|C'| = k$, is a power of 3. It is well known that the restriction of $[\delta^k]$ to C' is trivial. Thus $[\delta^k]$ is trivial, and then the order of $[\delta]$ is a power of 3. Therefore, $[\delta]$ is trivial and thus, δ is a coboundary. Now we conclude that φ is inner.

In the case that $p = 3$, we may assume that $\varphi|_{G_3} = id|_{G_3}$, $\varphi|_{G_2} = conj(b_2)|_{G_2}$ for some $b_2 \in supp(\alpha_0) \subset H_1$. Following the same line as that in the above, we can prove that φ is inner. Therefore, this completes the proof. □

Now we are ready to prove our main result (Theorem 2.1).

Since a G-adapted ring R is an integral domain of characteristic 0, we may assume that $\mathbb{Z}$ the ring of all rational integers is a subring of R. Because RA has only trivial units, $\mathbb{Z}A$ has only trivial units too.

So it follows from Higman's Theorem ([1], see also Theorem (2.7) in [15]) that A is an abelian group of exponent $2, 3, 4$ or 6. If $A = G/B$ is an abelian group of exponent $2, 4$ or 3, then $G = G_2$ or $G = G_3$. Therefore, the result follows from Lemma 2.2. In the case that A is an abelian group of exponent 6, the result follows from Lemma 2.3.

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